Bounded and Unbounded Currents in Nanoelectronic Circuits

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Abstract— **The application of emerging nanoelectronic tunneling devices, such as single-electron tunneling transistor, is only possible if a circuit theory exists for designing nanoelectronic circuits. One of the most important new phenomena that have to be included in the circuit descriptions is the quantum mechanical tunnel effect. It is modeled with an impulsive current source in nanoelectronics. When the tunnel condition is met, an electron tunneling through a tunnel junction TJ must be described by a tunnel current $i_{\mathbf{TJ}} = \mathrm{e}\delta(t - t_0)$, in which e is the elementary electron charge.**

One of the consequences of the description of tunneling with an impulsive tunnel current is that both unbounded and bounded currents may appear in the analysis of nanoelectronic circuits. Circuit analysis of such circuits involves besides a basic understanding of these currents, the inclusion of tunneling times, and stochastic behavior; uncertainty in the voltages in the circuit during tunneling will result.

The concepts can be illustrated and proved to describe the correct behavior by considering concrete circuits. This paper, therefore, focuses on the analysis of some concrete, but still simple, nanoelectronic circuits. First an introduction to bounded and unbounded currents is presented. Based on energy considerations, embedded in circuit theory by Tellegen's theorem, the unbounded tunnel current may or may not penetrate into the circuit exciting the tunnel junction. After this introduction, the most basic tunneling circuit, i.e., the single tunnel junction excited by an ideal voltage source will be fully explained in terms of current, voltage, and energy conservation in the circuit. Then, a real voltage source (a battery) is described. Again the circuit can be fully explained in terms of current, voltage, and energy conservation.

Keywords— **Nanoelectronics, single-electron tunneling devices, circuit theory, quantum devices, unbounded currents.**

I. INTRODUCTION

The concepts bounded and unbounded currents come from some considerations in network theory related to circuits including capacitors and switches (or, more general, stepping voltages). As such it is closely related to intriguing circuits, such as the switched capacitor circuit of Figure 1 that seems to violate energy conservation (Tellegen's theorem). It is also related to the description of energy in networks that have reduction in "free-energy" and thus to nanoelectronics, as discussed in this paper.

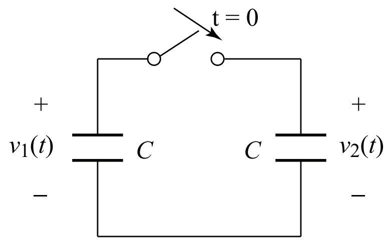

Fig. 1. *Switched capacitor network example. Assume an initial voltage across one of the capacitors and then close the switch; the energy stored on the capacitors will be halved. Note that at t=0 the voltages on both capacitors step.*

A. Some definitions

To describe circuits that include switches we assume that the switch opens (or closes) at $t = 0$, we will distinguish $t = 0^-$ just before opening (closing) the switch, and $t = 0^+$ immediately after opening (closing) the switch. Defined are:

$$f(0^+) \stackrel{\text{def}}{=} \lim_{t \downarrow 0} f(t) \tag{1}$$

and

$$f(0^-) \stackrel{\text{def}}{=} \lim_{t \uparrow 0} f(t) \tag{2}$$

In most textbooks on circuit or system theory models for switches are described using either operational calculus or Laplace transforms. For our purpose the models are best described using (Heaviside's) operational calculus, making it possible to describe the models in the time-domain. Used are the following definitions for the operators p and $1/p$:

$$p \stackrel{\text{def}}{=} \frac{d}{dt} \quad (3)$$

and

$$\frac{1}{p} \stackrel{\text{def}}{=} \int_{-\infty}^{t} ()d\alpha. \quad (4)$$

For a capacitor with a constant capacitance C holds:

$$i(t) = Cpv(t) \quad (5)$$

and

$$v(t) = \frac{1}{Cp} i(t) \quad (6)$$

Furthermore the following notation is used:
-the unit step function $\varepsilon(t)$ is defined by the relation

$$\varepsilon(t) \stackrel{\text{def}}{=} \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \quad (7)$$

At $t = 0$ the unit step function is undefined but restricted between 0..1.

-the delta function or impulse function $\delta(x)$ is defined as

$$\delta(t) \stackrel{\text{def}}{=} \begin{cases} 0 & t > 0 \\ \int_{-\infty}^{\infty} \delta(\tau)d\tau = 1 & t = 0 \\ 0 & t < 0 \end{cases} \quad (8)$$

The $\varepsilon(t)$ and $\delta(t)$ are not regular analytic functions but generalized functions. They define each other:

$$\int_{-\infty}^{t} \delta(\tau)d\tau = \varepsilon(t) \quad \text{or} \quad \frac{1}{p}\delta(t) = \varepsilon(t) \quad (9)$$

$$\frac{d}{dt}\varepsilon(t) = \delta(t) \quad \text{or} \quad p\varepsilon(t) = \delta(t) \quad (10)$$

B. Continuity property in linear networks

It is possible to predict whether a capacitor voltage $v_C(t)$ in a network (circuit) containing switches is *continuous*, that is if $v(0^+) = v(0^-)$ for this capacitor. In the circuit of the two capacitors (Figure 1), for example, we see that the capacitor voltages are discontinuous, if $v_1(0^-) \neq v_2(0^-)$. The following theorem predicts whether capacitor voltages are continuous in networks.

Theorem: [1] In a linear network consisting of resistors, inductors, capacitors, switches operating simultaneously, d.c. voltage sources and current sources, one is sure that: the capacitor voltages are continuous except in those capacitors which, together with at least one make contact, form a loop in the network derived from the original by reducing all source intensities to zero.

C. Continuity property of bounded capacitor currents

For convenience, only linear time-invariant capacitors are assumed. In general, using the constitutional relation $v = C^{-1}q$ and the definition of the current $i = dq/dt$ the voltage across a capacitor is given by:

$$v(t) = C^{-1}q(t_0) + C^{-1}\int_{t_0}^{t} i(\tau)d\tau \qquad t \geq t_0. \quad (11)$$

More specific, if $q(0^-)$ is the charge on a capacitor with capacitance C just before closing the switch the voltage across a capacitor is:

$$v(t) = C^{-1}q(0^-) + C^{-1}\int_{0^-}^{t} i(\tau)d\tau \qquad t \geq 0^-. \quad (12)$$

Clearly, to know the voltage across the capacitor we need to know the entire past history. To examine the continuity of the capacitor voltage we focus at $t = 0$, the moment of switching.

In defining bounded currents we follow Chua, Desoer and Kuh [2]. If the current $i_C(t)$ in a linear time-invariant capacitor is *bounded* in a closed interval $[0^-, 0^+]$, then the voltage $v_C(t)$ across the capacitor is a *continuous* function in the open interval $(0^-, 0^+)$. In particular, for time $t = 0$ satisfying $0^- < 0 < 0^+$ holds: $v_C(0^-) = v_C(0) = v_C(0^+)$.

Proof: Substituting $t = t_0 + \Delta t$ in Equation 11, where $t_a < t_0 < t_b$ and $t_a < t_0 + \Delta t \leq t_b$, we get:

$$v_C(t_0+\Delta t) - v_C(t_0) = C^{-1}\int_{t_0}^{t_0+\Delta t} i_C(\tau)d\tau \qquad t_0+\Delta t \geq t_0. \quad (13)$$

Since $i_C(t)$ is *bounded* in $[t_a, t_b]$, there is a *finite* constant M such that $|i_C(t)| < M$ for all t in $[t_a, t_b]$. It follows that the area under the curve $i_C(t)$ from t_0 to $t_0 + \Delta t$ is at most $M\Delta t$ (in absolute value), which tends to zero as $\Delta t \to 0$. Hence Equation 13 implies

[1] A.Henderson, Continuity of inductor currents and capacitor voltages in linear networks containing switches, Int. J. Electronics, 1971, Vol. 31, pp 579-587

[2] Chua, Desoer and Kuh, Linear and Nonlinear Circuits, 1987

that $v_C(t_0 + \Delta t) \rightarrow v_C(t_0)$ as $\Delta t \rightarrow 0$. This means that the voltage $v_C(t)$ is continuous at $t = t_0$. Filling in $t_a = 0^-$ and $t_b = 0^+$ the continuity for bounded currents at $t = 0$ is proved. (A continuity property also holds for inductor currents if the voltage across the inductor is bounded.)

D. *Unbounded currents*

In this context an unbounded current is an impulsive current; an unbounded current may appear in capacitor circuits with stepping ideal voltage sources in the absence of resistive elements. Consider for example an initially uncharged capacitor excited by an ideal voltage source. The voltage source waveform is a step function stepping at $t = 0$ from 0 to V_s Volt. The current through the circuit at $t = 0$ can be calculated by the constitutional relation of the capacitor, Equation 5,

$$i(t) = Cpv(t) = Cp[(V_s\epsilon(t)] = CV_s\delta(t). \tag{14}$$

CV_s is the charge transferred during the current pulse in this circuit.

E. *Voltages in circuits with unbounded currents*

A definition of the voltage across a capacitor *in the unbounded case*, can be found using Equation 12. If Δq is the charge transferred during the delta pulse then for $t > 0$ holds:

$$v(t) = C^{-1}q(0^-) + C^{-1}\Delta q \int_{0^-}^{t} \delta(\tau)d\tau \tag{15}$$

$$= C^{-1}q(0^-) + C^{-1}\Delta q. \tag{16}$$

Δq is a positive quantity if positive charge is entering a capacitor and negative if positive charge is leaving a capacitor.

II. SOLUTIONS OF THE TWO CAPACITOR PROBLEM

Tunneling is a quantum mechanical phenomenon. As quantum mechanics is described in terms of energies (Hamiltonians), energies will also play an important role in the description of nanoelelectronics circuits that use the tunnel phenomenon. Because those nanoelectronic circuits in some respect resemble the two capacitor circuit the study of energy conservation in this circuit—as mandated by Tellegen's theorem—is essential. For the energy problem of the capacitor circuit of Figure 1 there exists two solutions, the first considers only bounded currents, the second only unbounded currents.

A. *Solution A: bounded currents*

First, we have the standard textbook solution in which a resistor is placed between the two capacitors. By inserting the resistor the current in the circuit is bounded, the voltages across the capacitors will only change gradually, continuity of the capacitor currents is guaranteed, and the resistor absorbs the missing energy.

For example, assuming $v_2(0^-) = 0$ we can easily find the bounded current in the circuit:

$$i(t) = \frac{v_1(0^-)}{R} \exp\left(\frac{-2t}{RC}\right) \qquad t > 0 \tag{17}$$

and for the energy dissipated in the resistor w_R:

$$w_R(t) = \frac{Cv_1(0^-)^2}{4}\left[\exp\left(\frac{-4t}{RC}\right) - 1\right] \qquad t > 0. \tag{18}$$

We see that in steady-state the amount of energy dissipated in the resistor is $Cv_1(0^-)^2/4$, which is equal to the amount of energy stored on both capacitors *after* the switch is closed and half of the energy stored on capacitor 1 *before* the switch was closed. We see that energy is conserved at all times.

B. *Solution B: unbounded currents*

Second, there exists a solution in which the current is unbounded. This solution starts by representing charged capacitors by their equivalent initial condition models, in which the charged capacitor is modeled as an uncharged capacitor in series with a voltage source representing the charge on the capacitor. Because the change of charge on the capacitors will now be modeled as a step function in the initial conditions (from $v(0^-)$ to $v(0^+)$) the resulting current will be an impulse function—the current through the capacitors being proportional to the derivative of the voltage across them. For example, Davis [3] gives an excellent treatment.

For our circuit we find:

$$v(t) = \frac{v_1(0^-) + v_2(0^-)}{2}\varepsilon(t). \tag{19}$$

This voltage defines the final charge distribution for $t > 0$. And the current is

$$i(t) = Cp[v(t) - v_2(0^-)\varepsilon(t)] = \frac{1}{2}C[v_1(0^-) - v_2(0^-)]\delta(t).$$

[3] A.M. Davis, A Unified Theory of Lumped Circuits and Differential Systems Based on Heaviside Operators and Causality, IEEE Transaction on Circuits and Systems—1: Fundamental Theory and Applications, Vol. 41, 1994, pp:712-727

(20)

This current is the current redistributing the charge at $t = 0$.

Again we can consider the case that the initial charge on the second capacitor is zero. We see from Equation 20 that half of the initial charge on the first capacitor is transported to the second with an impulsive current $i(t) = (1/2)Cv_1(0^-)\delta(t)$.

Now, just as in the first solution we are going to consider the energy in this circuit: what is dissipating half of the energy in the absence of a resistor?

C. Energy generation in circuits with unbounded currents

To understand what dissipates the energy in a circuit with unbounded currents, I first describe energy generation and the energy balance in the simplest circuit involving an impulsive current, namely, a single capacitor charged by an ideal voltage source. The circuit charges a initially uncharged capacitor with a charge q. The ideal voltage source, therefore, steps from 0 Volt to $C^{-1}q$ Volt at $t = 0$. An impulsive current will transfer an amount of q Coulomb towards the capacitor—and, of course, at the same time carry off an amount of ($-q$) from the negative side of the capacitor plates. The energy stored the capacitor is raised from 0 Joule to $q^2/(2C)$. We calculate the energy delivered by the source w_s:

$$w_s = \int vidt = C^{-1}q^2 \int \varepsilon(t)\delta(t)dt. \tag{21}$$

The integral can be calculated by partial integration and realizing that the delta function is the derivative of the step function we find:

$$w_s = \frac{q^2}{2C}. \tag{22}$$

We notice that, of course, energy stored on the capacitor is provided exactly by the source; energy is conserved in this circuit. Also note that we are now able to calculate the energy generated by a stepping voltage source at $t = 0$ through which a delta current pulse flows at $t = 0$. A more in-depth approach [4] reveals that if the step had been down the source would have been absorbing energy; this should not be a surprise, because the source is just a non-linear resistor being able to be both passive or active.

[4] J. Hoekstra, Single-Electron Tunneling Circuits, in: Fundamentals of Nanoelectronics, S. Buegel, M. Luysberg, K. Urban, R. Wasser (Eds.) Lecture manuscripts of the 34 IFF Spring School 2003, Schriften des Forschungszentrums Juelich Reihe Materie and Material/Matter and Materials, Vol. 14, ISBN 3-89336-319-X, 2003

D. Solution B: energy conservation

Now, we can attack the energy problem in case we have an unbounded current. The energy balance consists of energy storage before the switch is closed (that is, at $t = 0^-$), the energy storage after the switch is closed (that is, at $t = 0^+$), and the energy generated and/or dissipated *during* closing of the switch at $t = 0$. Again we consider for simplicity the charge on the first capacitor to be q and the charge on second capacitor to be zero. We find: the energy stored before closing is $q^2/(2C)$; the energy stored after closing is $2(q/2)^2/(2C) = q^2/(4C)$. For the energy generated and absorbed at $t = 0$ we realize that during closing a current $i(t) = (q/2)\delta(t)$ flew through the circuit while the voltage sources representing the initial conditions—in this case, the values at $t = 0^-$ and $t = 0^+$—were both stepping: the source of the first capacitor stepped down from qC^{-1} to $(q/2)C^{-1}$, and the source of the second capacitor stepped up from 0 to $(q/2)C^{-1}$. Using Equation 22 we calculate that the source of the first capacitor while stepping down *absorbed* an amount of energy equal to $(3/8)(q^2/C)$, while the source of the second capacitor while stepping up *generated* an amount of energy equal to $(1/8)(q^2/C)$. Consequently, the net result is that during switching the sources absorbed $q^2/(4C)$—that is, they dissipated the missing energy at $t = 0$!

E. Unbounded or bounded currents through circuits

We have seen that the energy delivered by a stepping voltage source for charging a capacitor with an unbounded current is:

$$w_s = \frac{q^2}{2C} \qquad \text{unbounded current.} \tag{23}$$

This is *fundamentally* different than in case a bounded current is used. To obtain a bounded current in this circuit a resistor must be added, in series with the capacitor, to limit the current. The energy delivered by a stepping voltage source in the latter case is

$$w_s(t) = \int i(t)C^{-1}q\varepsilon(t)dt = C^{-1}q \int i(t)dt \qquad t > 0. \tag{24}$$

In steady state $\int i(t)dt$ will be the total amount of transported charge q, so that we obtain in steady state:

$$w_{\mathrm{s}} = \frac{q^2}{C} \qquad \text{bounded current.} \tag{25}$$

The resistor will dissipate the difference between the amount of energy delivered by the source and the amount of energy stored by the capacitor (the proof of this can be found in many physics textbooks).

In fact the above described difference *proofs* that an unbounded current cannot flow (other than strictly local—we will find this in the description of the tunnel current in the next section) through a circuit in which resistors are present. The proof is thus based on energy arguments. Also the reverse is true: in a circuit including resistors an unbounded current cannot flow through the circuit.

III. SOME SINGLE-ELECTRON TUNNELING CIRCUITS

In nanoelectronics quantum-classical equivalent circuits exist for tunneling in metal-insulator-metal diodes and single-electron tunneling transistors [5]. The dominant one in a circuit synthesis context is the impulse model [6], it models the current representing a single tunneling electron by a impulsive current $i(t) = \mathrm{e}\delta(t - t_0)$—with e the elementary electron charge— in parallel with the junction capacitance C_{TJ}, and the critical voltage above which tunneling starts by hot-electron scheme: if τ is the transit time, that is, the total time for an electron to tunnel and to restore the thermal equilibrium distribution, then

$$t_0 \in [0, \tau] \quad \text{and}$$

$$\left\{v_{\mathrm{TJ}}^{\mathrm{cr}} = v_{\mathrm{TJ}}(t) : v_{\mathrm{TJ}}(t+\tau) = -v_{\mathrm{TJ}}(t)\right\}. \tag{26}$$

Now, we have enough tools to describe nanoelectronic circuits that include metallic single-electron tunneling junctions. I describe two of them.

A. TJ excited by an ideal voltage source

First we consider a metal-insulator-metal tunnel junction excited by a constant voltage source. For large capacitances the response is well-known and shows the *vi*-characteristic of a linear resistor at low source voltages. We assume it can be modeled with by a tunnel junction exited by an ideal constant voltage source, in addition we consider electrons to tunnel through the tunneling junction one-by-one. Figure 2 shows the circuit and the energy-band diagram if we follow a single electron. Due to the fact that our model

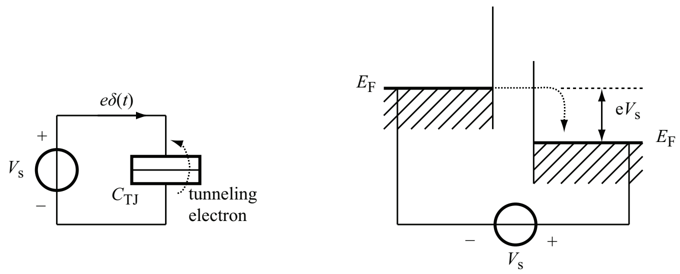

Fig. 2. *Single-electron tunneling junction excited by an ideal voltage source above the critical voltage and energy-band diagram showing the tunneling of a single electron from the Fermi level.*

circuit only consists of an ideal voltage source and a capacitor representing the junction's capacitance, but that resistors are absent, the unbounded impulsive tunnel current can propagate through the whole circuit. Analyzing this circuit we find:

$$\text{KVL:} \qquad v_{\mathrm{TJ}} = V_{\mathrm{s}} \tag{27}$$

$$\text{KCL:} \qquad i(t) = \mathrm{e}\delta(t) \tag{28}$$

Also the energy balance in the model circuit is well described. It is clear from the energy-band diagram that the tunneling electron, in this case, dissipates an amount of energy equal to $\mathrm{e}V_{\mathrm{s}}$ during the transition from the Fermi level at the negative side of the source to the Fermi level at the positive side of the source. The energy provided by the source is

$$\int V_{\mathrm{s}}\mathrm{e}\delta(t)dt = \mathrm{e}V_{\mathrm{s}} \int \delta(t)dt = \mathrm{e}V_{\mathrm{s}} \tag{29}$$

So, the source provides the dissipated energy at the time of tunneling. What, now, remains is to prove that the equivalent circuit for the tunneling junction also absorbs an amount of energy of $\mathrm{e}V_{\mathrm{s}}$.

If we consider the equivalent circuit to be a capacitor in parallel with a impulsive current source $i(t) = \mathrm{e}\delta(t)$ it can be proved [7]—the prove is developed in an analogues way as the prove of the energy generated by a stepping voltage source and a impulsive current—that the energy dissipated by this equivalent circuit is

$$w_{\mathrm{TJ}} = \frac{\mathrm{e}}{2}\left[v(0^-) + v(0^+)\right] \tag{30}$$

[5] A. I. Csurgay, On circuit models for quantum-classical networks, Int. J. Circ. Theor. Appl. 2007; 35:471-484

[6] J. Hoekstra, On Circuit Theories for Single-Electron Tunneling Devices, IEEE Trans. on Circuits and Systems, Nov. 2007.

[7] J. Hoekstra, Towards a circuit theory for metallic single-electron tunnelling devices, Int. J. Circ. Theor. Apply. 2007: 35:213-238.

In this case, where the voltage across the tunneling junction is constant, using Equation 30 filling in that $v(0^-) = v(0^+) = V_s$—we consider tunneling of a single electron at $t = 0$—we easily obtain the energy absorbed by the impulsive source to be eV_s too.

Two remarks must be made. First, as a consequence of the definition of the critical voltage and the conservation of energy, the inclusion of a tunneling time will result in a description in which the impulsive current will take place in the interval $[0, \tau]$. Second, to obtain a continuous current from the tunneling of single-electron, a Poisson distribution of the tunneling events has to be assumed.

B. TJ excited by a non-ideal voltage source

As might be clear from the previous discussions the modeling of the tunneling circuit with a real source, that is, with the inclusion of a resistor will prohibit the impulsive current to propagate through the model circuit. As a consequence Coulomb oscillation will appear, like in a circuit in which the tunneling junction is excited by a current source; also Coulomb blockade is feasible. As in this circuit all currents and voltages are bounded the validity of the Kirchhoff laws as well as energy conservation is always guaranteed. In this case the inclusion of a tunneling time will result in an uncertainty in the voltage across the tunnel junction, however, a state in which the voltage step and associated impulsive current exactly absorb the amount of energy to obtain balance in energy can always be found. To find the critical voltage the voltage before and after the tunnel event must be equal but opposite and the energy absorbed will be zero; because of the change in polarity the source will also not deliver any net energy.

IV. Conclusion

In this paper, I considered bounded and unbounded currents in the context of switched circuits and nanoelectronic single-electron tunneling circuits. Shown is that, first, energy is always conserved in these circuits—as is mandated by Tellegen's theorem. If energy is lost or minimized in final or steady state it just means that the missing energy was absorbed/dissipated during an unbounded current at $t = 0$. Some single-electron tunneling circuits have been analyzed fully to illustrate the theory.

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