Transition from Sinusoidal to Relaxation Oscillations in Emitter-Coupled Multivibrators

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***Abstract* — Method of quasilinear approximation is applied to investigate the sinusoidal oscillations obtained in emitter-coupled multivibrators. The amplitude and frequency are found, and frequency decreasing occurring in development of oscillation from the starting point to steady state is explained. The transition from sinusoidal to relaxation oscillations is explained by the modification of the phase plane partitioning when the coupling capacitance is varying. The jumps of the describing point in the phase plane help to understand the difficulties occurring in simulation of multivibrators. Finally, the formula for calculation of frequency in relaxation oscillations is given. All results are verified in simulations.**

***Index term*— Emitter-coupled multivibrators, sinusoidal oscillations, relaxation oscillations, van der Pol equations.**

I. INTRODUCTION

Emitter-coupled multivibrators are used very frequently in the systems requiring voltage-controlled oscillators, the phase-locked loops can be a typical, but not exhaustive, representative example. Forcing these circuits to oscillate at the maximal frequency (this is achieved reducing the coupling capacitance value) results in the oscillations which are more resembling to sinusoidal and not to relaxation oscillations. The transition from relaxation to sinusoidal oscillations (in the paper we move in the opposite direction) is continuous. The investigation of this process is important for using these multivibrators in high speed applications. .

It is known that classical Abraham-Bloch multivibrators at high frequencies may slip into sinusoidal regime. It was established experimentally long ago [1], and later on was explained analytically [2] using the method of energy cycles (we prefer to call it K.F. Theodorchik's method). Yet, the analytical treatment of similar transition in emitter-coupled multivibrators was never tried. Here we apply the method of [2] for consideration of sinusoidal oscillations developed in emitter-coupled multivibrator. We evaluate the amplitude and frequency of these oscillations. Then, using the phase plane partitioning we show the characteristics of the describing point movement that occur in the transition from sinusoidal to relaxation oscillations.

The structure of this paper is following. Section II gives the nonlinear differential equation describing the sinusoidal (more exactly, quasi-sinusoidal) oscillations in the emitter-coupled multivibrators. Section III describes the application of the method [2] to the differential equation obtained in Section II and results in amplitude transient equation. The amplitude and the frequency of the quasi-sinusoidal oscillations are found, and the limit cycle stability is verified. Section IV shows how the phase plane structure change in the transition from sinusoidal to relaxation oscillations. Section V gives some suggestions and describes simulation results. Section VI gives some conclusions.

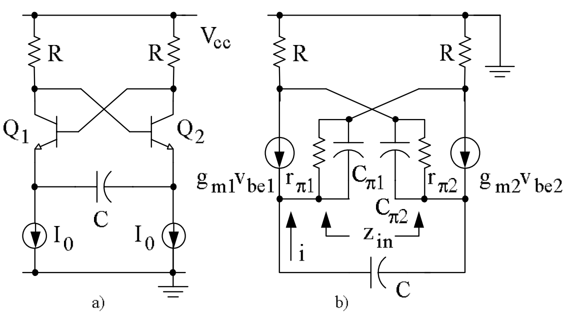

Fig. 1 Multivibrator and its small-signal circuit

II. SINUSOIDAL REGIME DIFFERENTIAL EQUATION

Fig. 1 shows the emitter-coupled multivibrator (Fig.1, a) and its simplified small-signal circuit (Fig. 1, b). Assuming sinusoidal oscillations and symmetric circuit (i.e. $r_{\pi1} = r_{\pi2} = r_\pi$, $C_{\pi1} = C_{\pi2} = C_\pi$ and $g_{m1} = g_{m2} = g_m$) one can find that the input impedance, z_{in} , connected to the coupling capacitor, C , is equal to

$$z_{in} = \frac{2[r_\pi + R(1 - r_\pi g_m) + R r_\pi C_\pi s]}{1 + r_\pi g_m + r_\pi C_\pi s} \quad . \tag{1}$$

Substituting (1) in the oscillation condition

$$i z_{in} + \frac{1}{C}\int i dt = 0 \quad , \tag{2}$$

and treating s as the time-domain differentiation operator one can obtain the differential equation for the capacitor current , i , as following

$$\frac{d^2 i}{dt^2}+\left[\frac{r_\pi\left(2+\frac{C_\pi}{C}\right)+2R(1-r_\pi g_m)}{2Rr_\pi C_\pi}\right]\frac{di}{dt}+\frac{(1+r_\pi g_m)}{2Rr_\pi CC_\pi}i=0 . \quad (3)$$

Usually $r_\pi g_m >> 1$, and (3) can be simplified to

$$\frac{d^2 i}{dt^2}+\left[\frac{1+(C_\pi/2C)-Rg_m)}{RC_\pi}\right]\frac{di}{dt}+\frac{g_m}{2RCC_\pi}i=0 . \quad (4)$$

When the oscillation amplitude is increasing the transistor transconductances g_{m1}, g_{m2} become the functions of current. They change in the opposite way (when one increases, the other decreases), and to include this current dependence we will substitute in (4)

$$g_m^{-1}\approx(g_{m1}^{-1}+g_{m2}^{-1}), \quad (5)$$

where $g_{m1}=(I_0+i)/V_T$ and $g_{m2}=(I_0-i)/V_T$. Then, using (5) and substituting it in (4) one obtains

$$\frac{d^2 i}{dt^2}+\left[\frac{\left(1+\frac{C_\pi}{2C}\right)-\frac{RI_0}{V_T}\left(1-\frac{i^2}{I_0^2}\right)}{RC_\pi}\right]\frac{di}{dt}+\frac{I_0\left(1-\frac{i^2}{I_0^2}\right)}{2RCC_\pi V_T}i=0 . \quad (6)$$

Introducing the normalized variable $x=i/I_0$ and using the notations $\delta_0=\frac{1}{2RC_\pi}\left(\frac{RI_0}{V_T}-\frac{C_\pi}{2C}-1\right)$, $\delta_2=\frac{I_0}{2C_\pi V_T}$, and $\omega_0^2=I_0/(2RCC_\pi V_T)$ one finally obtains

$$\frac{d^2x}{dt^2}-2(\delta_0-\delta_2x^2)\frac{dx}{dt}+\omega_0^2x(1-x^2)=0 . \quad (7)$$

III. AMPLITUDE TRANSIENT EQUATIONS

For the zero initial conditions $x(0)=(dx/dt)|_{t=0}=0$, the full solution of the differential equation

$$\frac{d^2x}{dt^2}+\omega^2x=\mathfrak{F}(t) \quad (8)$$

is equal to [2]

$$x=\frac{1}{\omega}\sin\omega t\int_0^t\mathfrak{F}(t)\cos\omega\xi d\xi-\frac{1}{\omega}\cos\omega t\int_0^t\mathfrak{F}(t)\sin\omega\xi d\xi . \quad (9)$$

Assume now that $\mathfrak{F}(t)=F(t)\cos\omega t$ where $F(t)$ is a function of time. Then (9) gives us

$$x=\frac{1}{2\omega}\sin\omega t\int_0^t F(\xi)d\xi+\frac{1}{2\omega}\left[\begin{array}{l}\sin\omega t\int_0^t F(\xi)\cos2\omega\xi d\xi\\ -\cos\omega t\int_0^t F(\xi)\sin2\omega\xi d\xi\end{array}\right] \quad (10)$$

If $F(t)=F_0=const$ the second term of (10) becomes zero. When $F(t)$ is not constant, this term will be much less than the first term, if $F(t)$ changes much slower than $\cos\omega t$. Hence, for slowly changing $F(t)$ the approximate solution of the equation

$$\frac{d^2x}{dt^2}+\omega^2x=F(t)\cos\omega t+f(t)\sin\omega t \quad (11)$$

will be

$$x=\frac{1}{2\omega}\sin\omega t\int_0^t F(\xi)d\xi-\frac{1}{2\omega}\cos\omega t\int_0^t f(\xi)d\xi , \quad (12)$$

i.e,

$$x=a(t)\sin\omega t-b(t)\cos\omega t , \quad (13)$$

where (approximately)

$$a(t)=\frac{1}{2\omega}\int_0^t F(\xi)d\xi;\quad b(t)=\frac{1}{2\omega}\int_0^t f(\xi)d\xi . \quad (14)$$

Differentiating (14) one obtains

$$\frac{da}{dt}=\frac{1}{2\omega}F(t);\quad \frac{db}{dt}=\frac{1}{2\omega}f(t) . \quad (15)$$

Let us now apply these results to obtain the solution of (7). We rewrite this equation as

$$\frac{d^2x}{dt^2}+\omega^2x=\omega^2x+2(\delta_0-\delta_2x^2)\frac{dx}{dt}-\omega_0^2x(1-x^2) \quad (16)$$

We assume that the solution of this equation can be represented as

$$x=A(t)\sin\omega t-B(t)\cos\omega t \quad (17)$$

and substitute (17) in the right side of (16). Yet, in doing so we consider that the amplitudes in (17) are changing so slowly that we may neglect their derivatives (this is not an error, it simply means that the mathematical foundation of the proposed approach lies in the averaging methods [3]). This gives us

$$\frac{d^2x}{dt^2}+\omega^2x=\left[\begin{array}{l}-\omega^2B+2\omega\delta_0A-\frac{2\omega\delta_2A(A^2+B^2)}{4}\\ +\omega_0^2B-\frac{3\omega_0^2B(A^2+B^2)}{4}\end{array}\right]\cos\omega t+\left[\begin{array}{l}\omega^2A+2\omega\delta_0B-\frac{2\omega\delta_2B(A^2+B^2)}{4}\\ -\omega_0^2A+\frac{3\omega_0^2A(A^2+B^2)}{4}\end{array}\right]\sin\omega t+... \quad (18)$$

Then, using (11), (13) and (15) one can write the amplitude transient equations (we prefer to call them van der Pol equations)

$$\begin{array}{l}\frac{dA}{dt}=\frac{1}{2\omega}\left[\begin{array}{l}-\omega^2B+2\omega\delta_0A-\frac{2\omega\delta_2A(A^2+B^2)}{4}\\ +\omega_0^2B-\frac{3\omega_0^2B(A^2+B^2)}{4}\end{array}\right]\\ \frac{dB}{dt}=\frac{1}{2\omega}\left[\begin{array}{l}\omega^2A+2\omega\delta_0B-\frac{2\omega\delta_2B(A^2+B^2)}{4}\\ -\omega_0^2A+\frac{3\omega_0^2A(A^2+B^2)}{4}\end{array}\right]\end{array} . \quad (19)$$

IV. ANALISYS OF TRANSIENT EQUATIONS

The steady-state solutions of (19) are obtained equating the derivatives to zero. This results in two mutually excluding solutions. The first one is $B=0$, and the amplitude A and frequency ω are obtained from the system of equations

$$\begin{cases} 2\delta_0\omega - 2\delta_2\omega\dfrac{A^3}{4} = 0 \\ \omega^2 A - \omega_0{}^2 A + \omega_0{}^2\dfrac{3A^3}{4} = 0 \end{cases}. \qquad (20)$$

The second solution is $A=0$, and amplitude and frequency that are obtained from the system that is the same as (20), and where A is replaced by B, and the signs of all terms in the second equation are opposite. Hence, as soon as oscillation starts, one solution is developed (let it be the first one) and the other is suppressed. One finds from (20) that

$$A = A_0 = 2\sqrt{\frac{\delta_0}{\delta_2}}; \quad \omega = \omega_0\sqrt{1-3\frac{\delta_0}{\delta_2}}. \qquad (21)$$

Hence, the oscillation initial frequency is ω_0, and when the oscillation arrives to steady state its frequency is reduced. The results (21) could be obtained by the harmonic balance method [3] as well.

Considering that the steady state solution is given by (21) and $B=0$, we are left with equation

$$\frac{dA}{dt} = \frac{1}{2\omega}\left[2\omega\delta_0 A - \frac{2\omega\delta_2 A^3}{4}\right]. \qquad (22)$$

Substituting $A = A_0 + \Delta A$ in (25) and doing linearization one finds that the perturbation, ΔA, of the limit cycle amplitude is described by the differential equation

$$\frac{d\Delta A}{dt} = -2\delta_0\Delta A. \qquad (23)$$

The characteristic equation of (23) has one negative root $p=-2\delta_0$. Hence, the oscillation is stable, and its stability increases when δ_0 is increasing, i.e. when the multivibrator moves from sinusoidal to relaxation oscillations.

Strictly speaking there is no pure sinusoidal regime in the multivibrators (as in any other oscillator as well), including the considered one. When the circuit moves to the relaxation regime, the oscillation becomes reach in harmonics [4]. The oscillation, first, starts to include the third harmonic. Using the method of [5] one can find that the development of the third harmonic increases the stability of the first harmonic.

V. RELAXATION OSCILLATIONS

When the ratio $C_\pi/(2C) \to 0$, the oscillator moves from nearly sinusoidal to relaxation oscillations. This transition can be traced using the phase plane. Using (6), and substituting the normalized variable $x = i/I_0$, one introduces $y = \dfrac{dx}{dt}$. Then (6) may be rewritten as

$$\frac{dy}{dx} = -\frac{\left[1+\dfrac{C_\pi}{2C} - \dfrac{RI_0}{V_T}(1-x^2)\right]y + \dfrac{I_0}{2CV_T}x(1-x^2)}{yRC_\pi}. \qquad (24)$$

One can see that the point $x=0; y=0$ is an unstable focus. The borders of increment and decrement regions can be obtained from the equation

$$1+\frac{C_\pi}{2C} - \frac{RI_0}{V_T}(1-x^2) = 0. \qquad (25)$$

This gives

$$x_b = \pm\sqrt{1-\left(1+\frac{C_\pi}{2C}\right)\frac{V_T}{I_0 R}}. \qquad (26)$$

The isocline of horizontal tangents can be obtained from the equation

$$y_z = -\frac{I_0}{2CV_T}\frac{x(1-x^2)}{\left[1+\dfrac{C_\pi}{2C} - \dfrac{I_0 R}{V_T}(1-x^2)\right]}. \qquad (27)$$

This gives three branches. The results may be summarized the following way (Fig. 2).

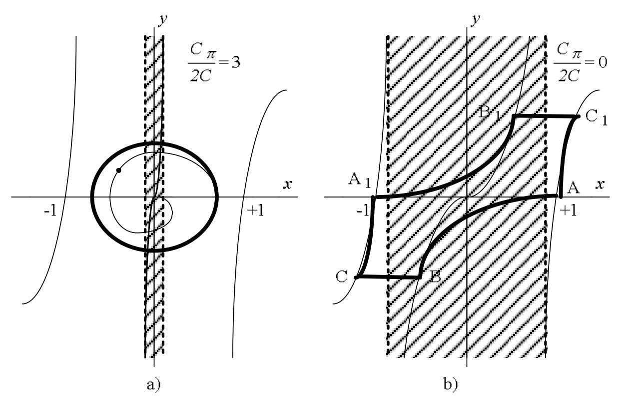

Fig. 2 Phase-plane partitioning

When $C_\pi/(2C)$ is large, the incremental zone (hatched region) is narrow. The oscillation starts within the incremental zone. Then the describing point leaves the incremental zone and starts to move in the decremental zones as well. Finally it arrives to the limit cycle (Fig. 2,a). The limit cycle includes two small parts where the oscillator absorbs energy, and two longer parts where the oscillator loses energy. The oscillator frequency, hence, in the steady-state condition should be less than the initial frequency. The limit cycle crosses the central isocline of horizontal tangents at the straight angle. The limit cycle is nearly a circle, and the oscillation shape is close to a sinusoidal one.

When $C_\pi/(2C) \to 0$, then, in accordance with (24) the describing point is able to move with high accelerations, i.e. it is able now to move by jumps in the phase plane. In addition, this point is obliged now to move very close (nearly along) to the isoclines of horizontal tangents, or along the line $y=0$. The phase plane now includes a wide incremental zone (Fig. 2,b). In the steady state oscillation, the describing point most of the first semi-period is sitting in, say, $x=1$ (point A), then it starts to move to the border of incremental zone, in the vicinity of the horizontal axis. As soon as it enters the incremental zone it will be in a very short time on the isocline of horizontal tangents, and arrives to the point B. Here it makes a jump on another branch of the isocline of horizontal tangents, to the point C, then quickly arrives to A_1. The first semi-period is finished. Another semi-period is similar: i.e. for most of the time the describing point is sitting in the point A_1, then moves to the border of incremental zone, enters it, quickly arrives to B_1, has a jump on another branch of the isocline of horizontal tangents, to the point C_1, and then quickly arrives to A.

This behaviour explains very well: a) the difficulty of some simulators to simulate the multivibrators when C_π is removed from the transistor model, and b) the appearance of very sharp and fast spikes in the current (seen at the collector resistances) because the point C_1 is further from the y-axis than A (or C is further from y-axis than A_1). In reality, the isocline of horizontal tangents is a continuous line going from a local maximum to the minimum (or vice versa) at the borders of incremental zone, and the jump from B_1 to C_1 or from B to C will be at the points of such local maximums or minimums.

Hence, the movement of the describing point from A (or from A_1) to the incremental zone is the only time when noise or other interference signal can interact with the multivibrator. It is very difficult to say when the describing point moved sufficiently off the line $x=0$ so that the accelerating properties of the phase plane will bring it to another point of rest. Yet, this time is important for calculation of the oscillation frequency. A good result gives the following approach.

When one transistor is ON and another is just start to operate the resistor seen at the capacitor terminals is

$$z_{in1}(0)=\frac{1}{g_{m1}}+\frac{1}{g_{m2}}-2R=\frac{V_T}{2I_0-\Delta I}+\frac{V_T}{\Delta I}-2R\,. \qquad (28)$$

Let us consider that the describing point is sufficiently moved from the horizontal axis when this value becomes equal to $-R$. This gives us

$$\Delta I = I_0(1-a)\,, \qquad (29)$$

where $a=\sqrt{1-\frac{2V_T}{I_0R}}$. The voltage at the capacitor at this instant is equal to

$$V_c = 2I_0R\left[a-\frac{V_T}{2I_0R}\ln\left(\frac{1+a}{1-a}\right)\right]. \qquad (30)$$

Assuming now that the capacitor is recharging from V_C to $-V_C$ by the current I_0 one finds the oscillation frequency as

$$f_o=\frac{1}{8RC\left[a-\frac{V_T}{2I_0R}\ln\left(\frac{1+a}{1-a}\right)\right]} \qquad (31)$$

This formula gives better results than calculating the frequency considering the capacitor as the short circuit, finding the loop transfer function, and equating it to 1 [6].

VI. Simulation Results

The essential simplification of the transistor model allows one to analyze the sinusoidal and relaxation regimes of the emitter-coupled multivibrator. Yet, it results in some difference between simulation results and the results given in the paper. The oscillation frequency ω_0 in simulations is lower by approximately 20%. A similar error occurs for the final frequency of ω. The results are improved, if the capacitance due to Miller effect of C_μ, the capacitor between collector and base, is added to C_π.

VII. Conclusion

The theory of energy cycles (we called it here “method of quasilinear approximation”) developed by K.F. Theodorchik allows one to solve rather complicated problems of the oscillations theory in applications to microelectronics. We demonstrated it by investigating the transition from sinusoidal to relaxation oscillation in the emitter-coupled multivibrator. The results obtained show that the sinosidal regime may be used in pushing the circuit to the maximal available speed, even though this results in a less stable limit cycle.

The method can be also expanded on the system of coupled oscillators. This expansion is now under development.

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