Evolution of the vortex part of the orbital angular momentum in separable first-order optical systems

Martin J. Bastiaans[1] and Tatiana Alieva[2]
[1]Technische Universiteit Eindhoven, Faculteit Elektrotechniek,
Postbus 513, 5600 MB Eindhoven, Netherlands.
Phone: +31 40 247 3319; Fax: +31 40 244 8375;
E-mail: m.j.bastiaans@tue.nl

[2]Universidad Complutense de Madrid, Facultad de Ciencias Físicas,
Ciudad Universitaria s/n, 28040 Madrid, Spain.
Phone: +34 91 394 4678; Fax: +34 91 394 4683;
Email: talieva@fis.ucm.es

Abstract—**We analyze the evolution of the vortex part of the orbital angular momentum during its propagation through separable first-order optical systems. We obtain that the evolution of the vortex part depends only upon the parameters a_x, a_y, b_x, and b_y of the ray transformation matrix, and that isotropic systems with the same ratio b/a produce the same change of the vortex part of the orbital angular momentum. Finally, it is shown that when light propagates through an optical fiber with a quadratic refractive index profile, the vortex part of the orbital angular momentum cannot change its sign more than four times per period.**

Keywords—**Wigner distribution, phase space signal description, optical beam characterization, partially coherent light, orbital angular momentum, vortex, twist**

I. INTRODUCTION

During the last decade the concept of the orbital angular momentum (OAM) has been applied for the description of coherent optical vortex beams [1,2]. In recent publications [3, 4] it has been suggested to decompose the OAM into two parts: the asymmetrical OAM and the vortex OAM. The first part describes an astigmatic beam but with a smooth wave front, while the second one is related to the singularity of the wave front (screw dislocations).

In this contribution we study the evolution of the asymmetrical OAM and vortex OAM of linear polarized, partially coherent beams during their propagation through separable first-order optical – also called $ABCD$ – systems.

It has recently been reported that some partially coherent fields also exhibit the vortex behavior [5]. Taking into account that the concept of the OAM can be generalized to the case of partially coherent beams [6], the results of this study can be applied to both the completely coherent and the partially coherent case.

II. WIGNER DISTRIBUTION MOMENTS AND ORBITAL ANGULAR MOMENTUM

Let partially coherent light be described by a temporally stationary stochastic process $f(x, y; t)$; as far as the time dependence is concerned, the ensemble average of the product $f(x_1, y_1; t_1) f^*(x_2, y_2; t_2)$, where the asterisk denotes complex conjugation, is then only a function of the time difference $t_1 - t_2$:

$$\mathrm{E}\left\{f(x_1, y_1; t_1) f^*(x_2, y_2; t_2)\right\} = \gamma(x_1, y_1, x_2, y_2; t_1 - t_2). \tag{1}$$

The function $\gamma(x_1, y_1, x_2, y_2; \tau)$ is known as the mutual coherence function [7–10] of the stochastic process $f(x, y; t)$. The mutual power spectrum [9, 10] or cross-spectral density function [11] $\Gamma(x_1, y_1, x_2, y_2; \omega)$ is defined as the temporal Fourier transform of the mutual coherence function:

$$\Gamma(x_1, y_1, x_2, y_2; \omega) = \int_{-\infty}^{\infty} \gamma(x_1, y_1, x_2, y_2; \tau)\, e^{j\omega\tau}\, d\tau. \tag{2}$$

For $x_1 = x_2 = x$, $y_1 = y_2 = y$, the cross-spectral density function reduces to the (auto) power spectrum $\Gamma(x, y, x, y; \omega)$, which represents the intensity distribution of the light for the temporal frequency ω. Since in the present discussion the explicit temporal-frequency dependence is of no importance, we shall, for the sake of convenience, omit the temporal-frequency variable ω from the formulas in the remainder of the paper.

A. Wigner distribution and its moments

The Wigner distribution [12, 13] of partially coherent light is defined in terms of the cross-spectral density function by

$$W(x,u;y,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma(x+\tfrac{1}{2}x',y+\tfrac{1}{2}y',\\ x-\tfrac{1}{2}x',y-\tfrac{1}{2}y')\,e^{-j2\pi(ux'+vy')}\,dx'\,dy'. \quad (3)$$

A distribution function according to definition (3) was first introduced in optics by Walther [14, 15], who called it the generalized radiance. The WD $W(x,u;y,v)$ represents partially coherent light in a combined space/spatial-frequency domain, the so-called phase space, where u is the spatial-frequency variable associated to the space variable x, and v the spatial-frequency variable associated to the space variable y.

The treatment in this paper is based on the normalized moments of the WD, where the normalization is with respect to the total energy E of the signal:

$$E=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W(x,u;y,v)\,dx\,du\,dy\,dv. \quad (4)$$

These normalized moments μ_{pqrs} of the WD are thus defined by

$$\mu_{pqrs}E=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W(x,u;y,v)\\ \times\, x^p u^q y^r v^s\,dx\,du\,dy\,dv \qquad (p,q,r,s\geq 0). \quad (5)$$

In the present paper we restrict ourselves to the ten second-order moments, $p+q+r+s=2$, which are usually combined into a positive-definite symmetric 4×4 matrix

$$\boldsymbol{M}=\begin{bmatrix}\mu_{2000} & \mu_{1010} & \mu_{1100} & \mu_{1001}\\ \mu_{1010} & \mu_{0020} & \mu_{0110} & \mu_{0011}\\ \mu_{1100} & \mu_{0110} & \mu_{0200} & \mu_{0101}\\ \mu_{1001} & \mu_{0011} & \mu_{0101} & \mu_{0002}\end{bmatrix}=\begin{bmatrix}\boldsymbol{R} & \boldsymbol{P}\\ \boldsymbol{P}^t & \boldsymbol{Q}\end{bmatrix},$$

where $\boldsymbol{R}$, $\boldsymbol{Q}$, and $\boldsymbol{P}$ are the 2×2 submatrices composed of the pure space moments, the pure spatial-frequency moments, and the mixed space/spatial-frequency moments, respectively. Moreover, without loss of generality, we assume that the four first-order moments μ_{1000}, μ_{0100}, μ_{0010}, and μ_{0001} are zero.

B. Orbital angular momentum

The OAM Λ of an optical beam can be expressed in terms of the second-order moments of its WD as [4, Eq. (3)]

$$\Lambda=\mu_{1001}-\mu_{0110}; \quad (6)$$

in comparison to [4], a mere constant in which the total energy of the light and the speed of light appear, has been omitted for the sake of convenience. The asymmetrical part Λ_a and the vortex part Λ_v of the OAM, can then be written as [4, Eqs. (22) and (21)]

$$\begin{aligned}\Lambda_a &= [(\mu_{2000}-\mu_{0020})(\mu_{1001}+\mu_{0110})\\ &\quad -2\mu_{1010}(\mu_{1100}-\mu_{0011})]/(\mu_{2000}+\mu_{0020}) \quad (7)\\ \Lambda_v &= 2\,[\mu_{0020}\mu_{1001}-\mu_{2000}\mu_{0110}\\ &\quad +\mu_{1010}(\mu_{1100}-\mu_{0011})]/(\mu_{2000}+\mu_{0020}), \quad (8)\end{aligned}$$

respectively; note that we have $\Lambda_a+\Lambda_v=\Lambda$. We remark that the vortex part Λ_v is closely related to the optical twist T, defined as [16]

$$T=\mu_{0020}\mu_{1001}-\mu_{2000}\mu_{0110}+\mu_{1010}(\mu_{1100}-\mu_{0011}). \quad (9)$$

Note that, whereas the OAM is related to the asymmetrical part of the submatrix $\boldsymbol{P}$, the twist is related to the asymmetrical part of the matrix product [16]

$$(\det\boldsymbol{R})\,\boldsymbol{R}^{-1}\boldsymbol{P}\\ =\begin{bmatrix}\mu_{0020} & -\mu_{1010}\\ -\mu_{1010} & \mu_{2000}\end{bmatrix}\begin{bmatrix}\mu_{1100} & \mu_{1001}\\ \mu_{0110} & \mu_{0011}\end{bmatrix}.$$

Hence, with

$$\boldsymbol{J}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},$$

we can write

$$\begin{aligned}\Lambda &= \mathrm{Sp}\,(\boldsymbol{PJ}), \quad (10)\\ \Lambda_v &= 2\,\frac{(\det\boldsymbol{R})\,\mathrm{Sp}\,(\boldsymbol{R}^{-1}\boldsymbol{PJ})}{\mathrm{Sp}\,(\boldsymbol{R})}=2\,\frac{\mathrm{Sp}\,(\boldsymbol{R}^{-1}\boldsymbol{PJ})}{\mathrm{Sp}\,(\boldsymbol{R}^{-1})}. \quad (11)\end{aligned}$$

C. Example: Partially coherent Gaussian light

As an example we consider the most general, partially coherent Gaussian light, which can be expressed by means of its cross-spectral density function as

$$\Gamma(x_1,y_1,x_2,y_2)=\Gamma(\boldsymbol{r}_1^t,\boldsymbol{r}_2^t)=\frac{1}{\pi}\sqrt{\det\boldsymbol{G}_1}\\ \times\exp\left(-\tfrac{1}{4}\begin{bmatrix}\boldsymbol{r}_1+\boldsymbol{r}_2\\ \boldsymbol{r}_1-\boldsymbol{r}_2\end{bmatrix}^t\begin{bmatrix}\boldsymbol{G}_1 & -i\boldsymbol{H}\\ -i\boldsymbol{H}^t & \boldsymbol{G}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{r}_1+\boldsymbol{r}_2\\ \boldsymbol{r}_1-\boldsymbol{r}_2\end{bmatrix}\right), \quad (12)$$

where we have chosen a representation that enables us to determine the Wigner distribution function of such light in an easy way. The exponent shows a quadratic form in which a 4-dimensional column vector $[(\boldsymbol{r}_1+\boldsymbol{r}_2)^t,(\boldsymbol{r}_1-\boldsymbol{r}_2)^t]^t$ arises, together with a 4×4 symmetric matrix. This matrix consists of four real 2×2 submatrices $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, $\boldsymbol{H}$, and $\boldsymbol{H}^t$, where, moreover, the matrices $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ (as well as the matrix $\boldsymbol{G}_2-\boldsymbol{G}_1$) are positive definite symmetric. The special form of the matrix is a direct consequence of the fact that the cross-spectral density is a nonnegative definite Hermitian function [10, 11].

In a more common way, the cross-spectral density of Gaussian light can be expressed in the form

$$\begin{aligned}\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2) = {} & \frac{1}{\pi}\sqrt{\det \boldsymbol{G}_1} \\ & \times \exp\{-\tfrac{1}{4}(\boldsymbol{r}_1-\boldsymbol{r}_2)^t(\boldsymbol{G}_2-\boldsymbol{G}_1)(\boldsymbol{r}_1-\boldsymbol{r}_2)\} \\ & \quad \times \exp\{-\tfrac{1}{2}\boldsymbol{r}_1^t[\boldsymbol{G}_1 - i\tfrac{1}{2}(\boldsymbol{H}+\boldsymbol{H}^t)]\boldsymbol{r}_1\} \\ & \quad \times \exp\{-\tfrac{1}{2}\boldsymbol{r}_2^t[\boldsymbol{G}_1 + i\tfrac{1}{2}(\boldsymbol{H}+\boldsymbol{H}^t)]\boldsymbol{r}_2\} \\ & \qquad\qquad \times \exp\{-\tfrac{1}{2}\boldsymbol{r}_1^t i(\boldsymbol{H}-\boldsymbol{H}^t)\boldsymbol{r}_2\}. \end{aligned} \tag{13}$$

Note that the asymmetry of the matrix $\boldsymbol{H}$ is a measure for the twist [17–19] of Gaussian light, and that general Gaussian light reduces to zero-twist Gaussian Schell-model light [20, 21], if the matrix $\boldsymbol{H}$ is symmetric, $\boldsymbol{H} - \boldsymbol{H}^t = \boldsymbol{0}$. In that case the light can be considered as spatially stationary light with a Gaussian cross-spectral density $\exp\{-\frac{1}{4}(\boldsymbol{r}_1-\boldsymbol{r}_2)^t(\boldsymbol{G}_2-\boldsymbol{G}_1)(\boldsymbol{r}_1-\boldsymbol{r}_2)\}$, modulated by a Gaussian modulator with modulation function $\exp\{-\frac{1}{2}\boldsymbol{r}^t(\boldsymbol{G}_1 - i\boldsymbol{H})\boldsymbol{r}\}$.

The moment matrix of Gaussian light takes the form

$$\boldsymbol{M} = \tfrac{1}{2}\begin{bmatrix} \boldsymbol{G}_1^{-1} & \boldsymbol{G}_1^{-1}\boldsymbol{H} \\ \boldsymbol{H}^t\boldsymbol{G}_1^{-1} & \boldsymbol{G}_2 + \boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{H} \end{bmatrix} \tag{14}$$

and hence

$$\begin{aligned} \boldsymbol{R} &= \tfrac{1}{2}\boldsymbol{G}_1^{-1}, \\ \boldsymbol{P} &= \tfrac{1}{2}\boldsymbol{G}_1^{-1}\boldsymbol{H}, \\ \boldsymbol{Q} &= \tfrac{1}{2}(\boldsymbol{G}_2 + \boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{H}). \end{aligned} \tag{15}$$

The OAM of such light thus reads

$$\Lambda = \mathrm{Sp}\,(\boldsymbol{PJ}) = \mathrm{Sp}\,(\boldsymbol{G}_1^{-1}\boldsymbol{HJ}) \tag{16}$$

and its vortex part can be expressed as

$$\Lambda_v = \frac{(\det \boldsymbol{R})\,\mathrm{Sp}\,(\boldsymbol{R}^{-1}\boldsymbol{PJ})}{\mathrm{Sp}\,(\boldsymbol{R})} = \frac{\mathrm{Sp}\,(\boldsymbol{HJ})}{\mathrm{Sp}\,(\boldsymbol{G}_1)}. \tag{17}$$

III. MOMENTS EVOLUTION IN SEPARABLE FIRST-ORDER OPTICAL SYSTEMS

It is well-known that the input-output relationship between the WD $W_{\text{in}}(x,u;y,v)$ at the input plane and the WD $W_{\text{out}}(x,u;y,v)$ at the output plane of a separable first-order optical system reads [22–24]

$$\begin{aligned} W_{\text{out}}(x,u;y,v) = {} & W_{\text{in}}(d_x x - b_x u, -c_x x + a_x u; \\ & d_y y - b_y v, -c_y y + a_y v). \end{aligned} \tag{18}$$

The coefficients a_x, b_x, c_x, d_x and a_y, b_y, c_y, d_y are the matrix entries of the symplectic ray transformation matrix [25] that relates the position x, y and direction u, v of an optical ray in the input and the output plane of the first-order optical system,

$$\begin{aligned} \begin{bmatrix} x_{\text{out}} \\ y_{\text{out}} \\ u_{\text{out}} \\ v_{\text{out}} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} x_{\text{in}} \\ y_{\text{in}} \\ u_{\text{in}} \\ v_{\text{in}} \end{bmatrix} \\ &= \begin{bmatrix} a_x & 0 & b_x & 0 \\ 0 & a_y & 0 & b_y \\ c_x & 0 & d_x & 0 \\ 0 & c_y & 0 & d_y \end{bmatrix} \begin{bmatrix} x_{\text{in}} \\ y_{\text{in}} \\ u_{\text{in}} \\ v_{\text{in}} \end{bmatrix}, \end{aligned} \tag{19}$$

where for separable system the four submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are diagonal. For separable systems, symplecticity reads simply $a_x d_x - b_x c_x = 1$ and $a_y d_y - b_y c_y = 1$. Note that in a first-order optical system, with such a symplectic ray transformation matrix, the total energy E, see Eq. (4), is invariant.

The normalized moments μ_{pqrs}^{out} of the output WD $W_{\text{out}}(x,u;y,v)$ are related to the normalized moments $\mu_{pqrs}^{\text{in}} = \mu_{pqrs}$ of the input WD $W_{\text{in}}(x,u;y,v)$ as

$$\begin{aligned} \mu_{pqrs}^{\text{out}} E = {} & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{\text{out}}(x,u;y,v) \\ & \qquad\qquad \times\ x^p u^q y^r v^s \, dx\, du\, dy\, dv \\ = {} & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{\text{in}}(d_x x - b_x u, \\ & -c_x x + a_x u; d_y y - b_y v, -c_y y + a_y v) \\ & \qquad\qquad \times\ x^p u^q y^r v^s \, dx\, du\, dy\, dv \\ = {} & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{\text{in}}(x,u;y,v) \\ & \times\ (a_x x + b_x u)^p (c_x x + d_x u)^q \\ & \times\ (a_y y + b_y v)^r (c_y y + d_y v)^s \, dx\, du\, dy\, dv \\ = {} & E \sum_{k=0}^{p}\sum_{l=0}^{q}\sum_{m=0}^{r}\sum_{n=0}^{s} \binom{p}{k}\binom{q}{l}\binom{r}{m}\binom{s}{n} \\ & \times\ a_x^{p-k} b_x^k c_x^l d_x^{q-l} a_y^{r-m} b_y^m c_y^n d_y^{s-n} \\ & \times\ \mu_{p-k+l,q-l+k,r-m+n,s-n+m}. \end{aligned} \tag{20}$$

We remark that for Fourier transforming systems, the output moments are related to the input moments by the simple relationships

$$\begin{aligned} \mu_{pqrs}^{\text{out}}\big|_{FT_x} &= \mu_{qprs}(-1)^q\, b_x^{p-q}, \\ \mu_{pqrs}^{\text{out}}\big|_{FT_y} &= \mu_{pqsr}(-1)^s\, b_y^{r-s}, \\ \mu_{pqrs}^{\text{out}}\big|_{FT_{x,y}} &= \mu_{qpsr}(-1)^{q+s}\, b_x^{p-q} b_y^{r-s}, \end{aligned} \tag{21}$$

where FT_x, FT_y, and $FT_{x,y}$, represent the Fourier transforms with respect to x (with parameters $a_x = d_x = 0$, $b_x c_x = -1$, $a_y = d_y = 1$, $b_y = c_y = 0$), to y (with parameters $a_x = d_x = 1$, $b_x = c_x = 0$, $a_y = d_y = 0$, $b_y c_y = -1$), and to both x and y (with parameters $a_x = d_x = 0$, $b_x c_x = -1$, $a_y = d_y = 0$, $b_y c_y = -1$), respectively. If we combine these simple relationships (21) with the definition of the twist (9), we can readily derive the twists $T^{\text{out}}|_{FT_x} = b_x T_x$, $T^{\text{out}}|_{FT_y} = b_y T_y$, and $T^{\text{out}}|_{FT_{xy}} = b_x b_y T_{xy}$ at the output of these respective Fourier transforming systems, and we have

$$\begin{aligned} T_x &= \mu_{0020}\mu_{0101} + \mu_{0200}\mu_{1010} \\ &\quad - \mu_{0110}(\mu_{1100} + \mu_{0011}), \\ T_y &= \mu_{0002}\mu_{1010} - \mu_{2000}\mu_{0101} \\ &\quad + \mu_{1001}(\mu_{1100} + \mu_{0011}), \\ T_{xy} &= \mu_{0002}\mu_{0110} + \mu_{0200}\mu_{1001} \\ &\quad - \mu_{0101}(\mu_{1100} - \mu_{0011}). \end{aligned} \tag{22}$$

IV. EVOLUTION OF THE TWIST AND THE ORBITAL ANGULAR MOMENTUM

There are seven relevant second-order moments which compose the vortex (7) and the asymmetrical (8) part of the OAM; the purely uv-moments (i.e., μ_{0200}, μ_{0002}, and μ_{0101}, for which with $p = r = 0$) do not enter the formulas (7) and (8). Combining these formulas with the moment relations (20), immediately leads to the following seven equations, which describe the values of these seven moments in the output of an $ABCD$ system in terms of the ten second-order input moments:

$$\mu_{2000}^{\text{out}} = a_x^2\mu_{2000} + 2a_xb_x\mu_{1100} + b_x^2\mu_{0200}, \tag{23}$$

$$\begin{aligned} \mu_{1010}^{\text{out}} &= a_xa_y\mu_{1010} + a_xb_y\mu_{1001} + b_xa_y\mu_{0110} \\ &\quad + b_xb_y\mu_{0101}, \end{aligned} \tag{24}$$

$$\mu_{0020}^{\text{out}} = a_y^2\mu_{0020} + 2a_yb_y\mu_{0011} + b_y^2\mu_{0002}, \tag{25}$$

$$\begin{aligned} \mu_{1100}^{\text{out}} &= a_xc_x\mu_{2000} + (a_xd_x + b_xc_x)\mu_{1100} \\ &\quad + b_xd_x\mu_{0200}, \end{aligned} \tag{26}$$

$$\begin{aligned} \mu_{0011}^{\text{out}} &= a_yc_y\mu_{0020} + (a_yd_y + b_yc_y)\mu_{0011} \\ &\quad + b_yd_y\mu_{0002}, \end{aligned} \tag{27}$$

$$\begin{aligned} \mu_{1001}^{\text{out}} &= a_xc_y\mu_{1010} + a_xd_y\mu_{1001} + b_xc_y\mu_{0110} \\ &\quad + b_xd_y\mu_{0101}, \end{aligned} \tag{28}$$

$$\begin{aligned} \mu_{0110}^{\text{out}} &= c_xa_y\mu_{1010} + c_xb_y\mu_{1001} + d_xa_y\mu_{0110} \\ &\quad + d_xb_y\mu_{0101}. \end{aligned} \tag{29}$$

In particular, the OAM Λ^{out} at the output of a separable $ABCD$ system takes the form

$$\begin{aligned} \Lambda^{\text{out}} &= \mu_{1001}^{\text{out}} - \mu_{0110}^{\text{out}} \\ &= \mu_{1001}(a_xd_y - b_yc_x) - \mu_{0110}(a_yd_x - b_xc_y) \\ &\quad + \mu_{0101}(b_xd_y - b_yd_x) + \mu_{1010}(a_xc_y - a_yc_x). \end{aligned} \tag{30}$$

In order to obtain the evolution of the vortex and asymmetrical parts of the OAM, let us first consider the output twist T^{out}. After a straightforward but rather lengthy calculation, using the definition of the twist (9), the moment relations (20), and the definitions (22), it can be shown that the following expression holds for the output twist:

$$T^{\text{out}} = a_xa_yT + a_yb_xT_x + a_xb_yT_y + b_xb_yT_{xy}. \tag{31}$$

We further need the expression

$$\begin{aligned} \mu_{2000}^{\text{out}} + \mu_{0020}^{\text{out}} &= a_x^2\mu_{2000} + 2a_xb_x\mu_{1100} + b_x^2\mu_{0200} \\ &\quad + a_y^2\mu_{0020} + 2a_yb_y\mu_{0011} + b_y^2\mu_{0002} \end{aligned} \tag{32}$$

for the denominator that appears in Eqs. (7) and (8). Note that the denominator is always positive, which is a direct consequence of the fact that the moments μ_{2000} and μ_{0200} (and also μ_{0200} and μ_{0002}) are positive by definition. From Eqs. (31) and (32), we immediately conclude that the evolution of the vortex part Λ_v through a separable $ABCD$ system depends only on the parameters a_x, a_y, b_x, and b_y of the ray transformation matrix. This also implies that Λ_v cannot be changed by a quadratic phase corrector ($a_x = a_y = 1, b_x = b_y = 0$); note that in this case $\Lambda^{\text{out}} = \Lambda + \mu_{1010}(c_y - c_x)$.

V. THE SPECIAL CASE OF AN ISOTROPIC SYSTEM

For an isotropic system, i.e., $a_x = a_y$, $b_x = b_y$, $c_x = c_y$, and $d_x = d_y$, the general equation (30) reduces to $\Lambda^{\text{out}} = \mu_{1001} - \mu_{0110} = \Lambda$ and we conclude that the OAM does not change when the beam propagates through an isotropic $ABCD$ system. However, this property does not hold for the vortex and asymmetrical parts of the OAM, as has been demonstrated for the case of free-space propagation [3,4]. In the special case of an isotropic system, Eqs. (31) and (32) reduce to

$$T^{\text{out}} = a^2T + ab(T_x + T_y) + b^2T_{xy}, \tag{33}$$

$$\begin{aligned} \mu_{2000}^{\text{out}} + \mu_{0020}^{\text{out}} &= a^2(\mu_{2000} + \mu_{0020}) \\ &\quad + 2ab(\mu_{1100} + \mu_{0011}) + b^2(\mu_{0200} + \mu_{0002}), \end{aligned} \tag{34}$$

respectively, and we thus have

$$\begin{aligned} \Lambda_v^{\text{out}} &= 2\left[a^2T + ab(T_x + T_y) + b^2T_{xy}\right] \\ &\quad \times \left[a^2(\mu_{2000} + \mu_{0020}) + 2ab(\mu_{1100} + \mu_{0011}) \right. \\ &\quad \left. + b^2(\mu_{0200} + \mu_{0002})\right]^{-1}. \end{aligned} \tag{35}$$

We remark that for the special signal for which the relation

$$\frac{T}{\mu_{2000} + \mu_{0020}} = \frac{T_x + T_y}{2(\mu_{1100} + \mu_{0011})} = \frac{T_{xy}}{\mu_{0200} + \mu_{0002}} \tag{36}$$

holds, the vortex part of its OAM does not change in isotropic systems. This holds in particular for rotationally symmetric beams, for which $\mu_{0020}=\mu_{2000}$, $\mu_{0002}=\mu_{0200}$, $\mu_{0011}=\mu_{1100}$, $\mu_{0110}=-\mu_{1001}$, and $\mu_{1010}=\mu_{0101}=0$, [26] and hence $T=2\mu_{2000}\mu_{1001}$, $T_x=T_y=2\mu_{1100}\mu_{1001}$, and $T_{xy}=2\mu_{0200}\mu_{1001}$, and for which the three expressions in Eq. (36) thus take the value μ_{1001}: a rotationally symmetric beam propagating through an isotropic system does not change its vortex and the asymmetrical part of its OAM remains zero.

We note that for isotropic systems with $b=0$ (and $ad=1$), the vortex part of the OAM (as well as the asymmetrical part of it) is preserved: $\Lambda_v^{\text{out}}=\Lambda_v$ and $\Lambda_a^{\text{out}}=\Lambda_a$. This is the case for all systems for which the input and output planes are conjugate planes, like a thin lens (with $a=d=1$, $b=0$, and c inverse proportional to the focal distance) and an ideal magnifier (with $a=m$, $d=1/m$, $b=c=0$). On the other hand, for isotropic Fourier transforming systems (with $a=0$ and $bc=-1$), the vortex part of the OAM is indeed changed according to

$$\begin{aligned}\Lambda_v^{\text{out}} &= 2\,\frac{T_{xy}}{\mu_{0200}+\mu_{0002}}\\ &= -2\,\frac{\mu_{0002}\mu_{0110}-\mu_{0200}\mu_{1001}+\mu_{0101}(\mu_{1100}-\mu_{0011})}{\mu_{0200}+\mu_{0002}}.\end{aligned} \tag{37}$$

Let us now study the general expression (35) for the evolution of the vortex part of the OAM, which we may as well express in the form

$$\begin{aligned}\Lambda_v^{\text{out}} = \Lambda_v(p) &= 2\,[T+p(T_x+T_y)+p^2T_{xy}]\\ &\quad\times\,[(\mu_{2000}+\mu_{0020})+2p(\mu_{1100}+\mu_{0011})\\ &\qquad+p^2(\mu_{0200}+\mu_{0002})]^{-1},\end{aligned} \tag{38}$$

where we have introduced the ratio $p=b/a$. Optical systems with the same p behave similarly with respect to the evolution of Λ_v. As examples we mention (i) a section of free space in the paraxial approximation, or 'parabolic' system [27] (with $a=d=1$, $c=0$, and b proportional to the propagation distance), (ii) a fractional Fourier transforming system [28], or 'elliptic' system [27] (with $a=d=\cos\alpha$ and $b=-c=\sin\alpha$), and (iii) a 'hyperbolic' system [27] (with $a=d=\cosh\alpha$ and $b=c=\sinh\alpha$), for which systems the parameter p corresponds to b, $\tan\alpha$, and $\tanh\alpha$, respectively. Analyzing Eq. (38), one can find the values of the parameter p where $\Lambda_v(p)$ is zero, or has maxima or minima. Note that, while the denominator is positive for all possible values of p, the numerator – and therefore $\Lambda_v(p)$ itself – may or may not change its sign (but not more than twice) depending upon the actual values of T, T_x+T_y, and T_{xy}. Clearly, $\Lambda_v(p)$ takes the value zero for those values of the parameter p for which $p=\{-(T_x+T_y)\pm[(T_x+T_y)^2-4TT_{xy}]^{1/2}\}/2T_{xy}$, and since p has to be real, this is possible only if the condition $(T_x+T_y)^2\geq 4TT_{xy}$ holds.

In the case of free-space propagation, for which p is associated with the propagation distance, only positive values of p are allowed, and $\Lambda_v(0)=(2E/c^2)T/(\mu_{2000}+\mu_{0020})$ and $\Lambda_v(p\to\infty)=(2E/c^2)T_{xy}/(\mu_{0200}+\mu_{0002})$; note that the case $p\to\infty$ corresponds to Fourier transforming. Hence, for $TT_{xy}<0$, there is only *one* positive solution where $\Lambda(p)$ changes its sign. On the other hand, for $TT_{xy}>0$, there are *two* positive solutions if the signs of T_{xy} and T_x+T_y are different, and there are *no* positive solutions if these signs are the same. Moreover, in the special case that the initial field (or its Fourier transform) is vortex free, $T=0$ (or $T_{xy}=0$), the vortex $\Lambda_v(p)$ takes the value zero also for $p=-(T_x+T_y)/T_{xy}$, which is positive if T_{xy} and T_x+T_y have different signs (or for $p=-T/(T_x+T_y)$, which is positive if T and T_x+T_y have different signs).

In the case of a fractional Fourier transforming system, for which p is associated with the fractional angle α, $p=\tan\alpha$, both positive and negative values of p are acceptable. Positive values of p correspond to the first and third quadrants of the angle α, and negative values to the second and fourth ones. Therefore, during one period $0\leq\alpha\leq 2\pi$, the sign of $\Lambda_v(p)$ changes four times if $(T_x+T_y)^2>4TT_{xy}$, twice if $(T_x+T_y)^2=4TT_{xy}$, and it does not change if $(T_x+T_y)^2<4TT_{xy}$. Note that this analysis is useful for understanding the propagation of vortex beams through optical fibers with quadratic refractive index profiles, since in the paraxial approximation of the scalar diffraction theory, the complex field amplitude of monochromatic light propagating through such a fiber is fractionally Fourier transformed, with the angle α being proportional to the propagation distance in the fiber [29, 30].

VI. CONCLUSIONS

We have obtained a general expression describing the evolution of the vortex part of the OAM in first-order optical systems, which contains only the parameters a_x, a_y, b_x, and b_y of the ray transformation matrix. In the case of isotropic systems, the evolution of the vortex and asymmetrical parts of the OAM is defined only by the parameter b/a.

The results of this paper can be used for the design of vortex beams by means of $ABCD$ systems and for the description of their propagation through such systems. In particular the propagation of the vortex part of the

OAM through free space and through an optical fiber with quadratic refractive index (fractional Fourier transforming system) has been discussed.

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