The Effect of Order-Statistics Filtering on the Output Probability Density Functions

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Abstract—**The SNR of a single Magnetic Resonance Spectroscopy (MRS) signal acquisition is usually insufficient for meaningful parameter estimation. Therefore acquisition is repeated M times and the resulting M signals are *averaged* in order to obtain an improved SNR (signal-averaging). In this work, an alternative to signal-averaging, *order-statistics filtering*, is used to improve SNR, and moreover, to eliminate motion-related signal artifacts. Whereas averaging of M signals results in one single SNR-improved output, order-statistics filtering results in M output signals which are all SNR-improved. The present work concerns computation of the probability density function of output noise of order-statistics filters, given a specific probability density function of the input noise from the MRS scanner.**

Keywords— **MR Spectroscopy, patient motion, artifacts, order-statistics filtering**

I. Introduction

Usually, the SNR of a signal obtained from a single Magnetic Resonance Spectroscopy (MRS) acquisition is low. In order to improve the SNR, signal-acquisition is repeated M times, and the resulting M signals are added (averaged). In a typical MRS scan, M can be 32.

Due to **patient motion** during acquisitions, a number of signals may contain outliers. In simple averaging such outliers are not eliminated. As a result, averaged signals may contain motion-artifacts that hamper proper parameter estimation.

However, averaging of multiply acquired noisy signals is not the only way to improve SNR. Order-statistics filtering is a viable alternative that offers additional possibilities such as testing the quality of the signal, *i.e.*, detecting absence or presence of outliers. One clear distinction with respect to averaging is that M output signals are produced, each ordered (ranked) into one of the M filter 'channels'. On the other hand, averaging produces only one output signal, in a single channel.

Special in this approach is that all M output signals can be analysed and even used for parameter estimation. Therefore we call it 'All-Rank Selection Order-Statistics filtering', abbreviated as ARSOS.

In this study we show that *statistical analysis* of the M output signals of ARSOS provides a measure of the quality of a signal. In particular, we study the effect order-statistics filtering on the shape of the probability density function (pdf) of the output signals, given the shape of the pdf of the input signals.

In MRS, order-statistics filtering was introduced for the first time in the study of Ref. [1], a successful attempt to eliminate signal artifacts. Subsequently, ARSOS appeared capable of strongly reducing the effects of patient-motion [2, 3]. It is applicable to so-called 'single-voxel' MRS.

The organisation of this paper can be gleaned from the Contents list and Figures list at the end.

II. Materials and Methods

A. Order-Statistics Filtering

Linear filtering techniques have serious limitations when dealing with signals that have been created or processed by a system exhibiting some degree of non-linearity [4]. Among the signals that linear filters perform poorly on are those with changing levels (e.g. due to patient motion) and corrupting noise that is either heavy tailed (which means that the signal contain outliers), or signal dependent.

Image processing is the field where non-linear filter techniques have first shown clear superiority over linear filters [5]. The design of non-linear filters can follow many approaches since there is no single underlying theory on non-linear filtering [6]. One non-linear filtering approach that has received considerable attention, and for which much theoretical study has been conducted, is that of so-called Rank-Order Filtering, a method whose filtering effect is obtained by rank-ordering the input data. Rank-order filtering is also known as Order-Statistics Filtering.

Note that in MRS context 'rank' stands for the value (size) of the sample. Sample values perturbed by patient motion will be shifted to far beyond the noise. Rank-order filtering will therefore push such samples toward outer regions of the output, far from the median value.

Much attention has been paid to rank-order filters since the 'running-median filter' was first applied to the smoothing of time series by Tukey in 1974 [7]. Rank-ordering of samples enables the design of filter structures that are (a) robust in environments where the assumed statistics deviate from Gaussian (which is the case in patient movement during data acquisition) and are possibly contaminated with outliers, and (b) track signal discontinuities without introducing transient or blurring artifacts as linear filters do.

B. Commonly known Order-Statistics Filters

Of the group of order statistics filters the median-filter is best known [4–6, 8, 9]. Less known representatives are the minimum and maximum filter, which returns the minimum respectively maximum value of a set of input values. All these filters change the input pdf in a different way, *i.e.*, their output pdfs are all different. This paper deals with the computation of these output pdfs.

C. Effect of Order-Statistics Filtering on the pdf

Let $X_1, X_2, .., X_M$ be samples of a continuous *random* variable. We assume that the pdf $f(x)$ of the scanner noise is given by:

$$f(x) = \frac{\exp(-x^2/2)}{\sqrt{2\pi}} \ . \tag{1}$$

Normally one is interested in the probability that the random variable is smaller than a certain given value x. For this one needs the cumulative distribution function $F(x)$ (cdf) given by:

$$F(x) = \int_{-\infty}^{x} \frac{\exp(-x^2/2)}{\sqrt{2\pi}} \ . \tag{2}$$

The order statistic $X_{(i)}$ is a random variable defined to be the i^{th}-largest of the set $\{X_1, X_2, .., X_M\}$. Therefore,

$$X_{(1)} \leq X_{(2)} \leq X_{(M)} \ . \tag{3}$$

The cumulative distribution function for the i^{th} order statistic is denoted by $F_{X_i(x)} = \text{Prob}[X_{(i)} < x] = \text{Prob}[i$ or more observations are smaller than $x]$: and is given by [10]:

$$F_{X_{(i)}}(x) = \sum_{j=i}^{n} \binom{n}{j} [F(x)]^j [1 - F(x)]^{n-j} \ . \tag{4}$$

The associated pdf can be obtained by differentiation of Eq. (4) and is given by:

$$f_{X_{(i)}}(x) = n\binom{n-1}{i-1}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x) = \frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1}f(x)[1-F(x)]^{n-i} \ . \tag{5}$$

Note that the pdfs of the filtered output signals are not Gaussian anymore, and as can be seen in Fig.1, are asymmetric. The fact that the pdfs have asymmetric non-Gaussian shape may have a potential negative impact on (non-linear) least-squares parameter estimation of such signals, since the performance of L_2-norm based methods deteriorate in non-stationary or not Gaussian noise environments especially when the data contains outliers [11].

Intuitively, we expect that estimating parameters from order-statistics filtered signals using the least-squares criterion, traditionally applied in MRS, leads to bias [12]. For instance, the simple case of estimating the value of some constant signal plus added noise whose pdf has a positive skewness [13], will come out too large, *i.e.*, be biased. We expect that more complicated cases of MRS-parameter estimation based on the least-*squares* criterion result in bias too.

D. Combining Output-Channels of an Order-Statistics Filter

This Section shows that if the pdf of the input signal of an ARSOS filter is symmetric, appropriate pairing of asymmetric output signals can restore symmetry. Specifically, one can show that:

$$f_{X_{(i)}}(x) = f_{X_{(n-i+1)}}(x) \ . \tag{6}$$

Eq. (6) can be proved in the following way. First, we write

$$f_{X_{(n-i+1)}}(x) = \frac{n!}{(n-i)!(i-1)!}[F(x)]^{n-i}f(x)[1-F(x)]^{i-1} \ . \tag{7}$$

Furthermore, $F(x)$ is symmetric with respect to the point (0,½), leading to the equality

$$F(x) = 1 - F(-x) \ . \tag{8}$$

Together with $f(x) = f(-x)$ we then substitute Eq. (8) in Eq. (6) and get

$$f_{X_{(n-i+1)}}(x) = \frac{n!}{(n-i)!(i-1)!}[1-F(-x)]^{n-i}f(-x)[F(-x)]^{i-1} \ , \tag{9}$$

from which follows

$$f_{X_{(n-i+1)}}(-x) = \frac{n!}{(n-i)!(i+1)!}[1-F(x)]^{n-i}f(x)[F(x)]^{i-1} = f_{X_{(i)}}(x) \ . \tag{10}$$

III. Results and Discussion

A. Example: Output-pdfs of an ARSOS[3] filter

To provide an idea of how output pdfs look, we computed them for an ARSOS[3] filter using Eq. (5) and assuming a Gaussian input pdf.

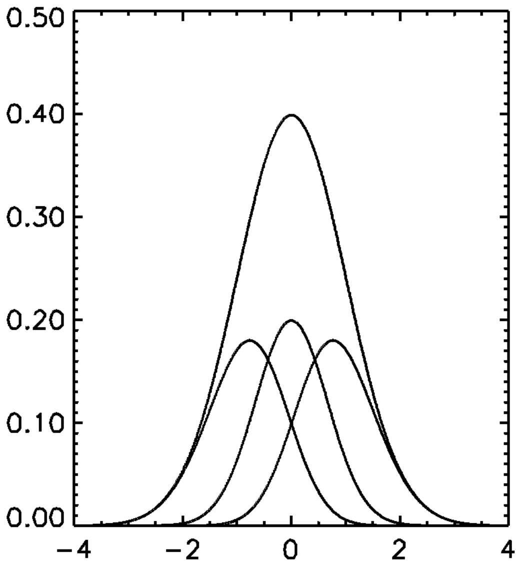

Fig. 1

The large curve is the pdf of the standard normal distribution $N(0,1)$ given by Eq. (1). The smaller curves are the output pdfs of the three output signals $f_{X_{(i)}}(x)$ of an ARSOS[3] filter. Note the shifts of the centres of two pdfs, which are indicative of asymmetry.

Fig. 1 displays the three output pdfs of the ARSOS[3] filter, and Fig. 2 shows the corresponding cdf. The large curve in Fig. 1 is the pdf of the standard normal distribution ($N(0,1)$, Eq. (1)) which was the input to the filter and the pdfs of the three outputs of ARSOS[3] filter given by Eq. (5).

As can be seen from Eq. (5), the output pdfs of the filter are not Gaussian anymore. Except for the median (middle of the three), all pfds are (slightly) asymmetric. Note that the small curve on the left is the pdf belonging to the minimum filtered output, the curve in the middle is the pdf of the median and the curve to the right is the pdf of the maximum filtered output. Also note that the widths of output pdfs are smaller than those of the input distribution. This means that, when applied to a real-world signal comprising a deterministic and a noise part, the SNR improves. For the median output, the SNR reaches its optimal value, and is given by [6].

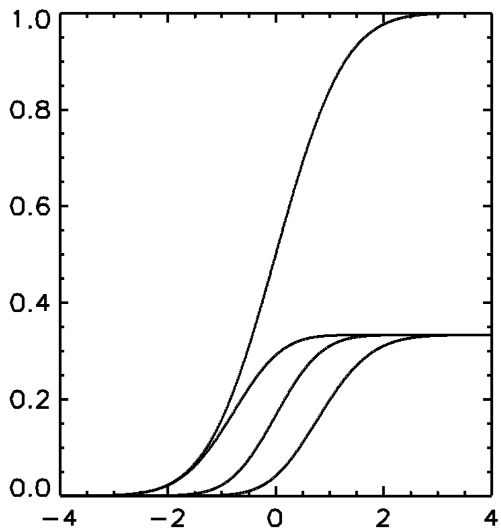

Fig. 2

Same as displayed in Fig. 1 but the cumulative distribution functions of three filter outputs.

$$\sigma^2_{\text{out}} = \frac{\pi\sigma^2_{\text{in}}}{2M} \, . \tag{11}$$

Eq. (11) implies that, as with conventional signal averaging, the application of median filtering too improves the SNR. For conventional averaging of M input signals the well-known relation

$$\sigma^2_{\text{out}} = \frac{\sigma^2_{\text{in}}}{M} \tag{12}$$

holds, which is consistent with the familiar fact that the SNR is proportional with the square root of the number of averaged acquisitions, *i.e.*, with $\sqrt{M}$. Comparing Eq. (11) with Eq. (12), one can directly see that median filtering performs 2dB less well than averaging.

B. Effect of ARSOS on Moments about the Mean

A random variable can be characterised by its standardised moments about the mean, defined as μ_k/σ^k [14]. The first moment is called the mean, the second moment the variance, the third moment the skewness, and the fourth moment the kurtosis. The skewness is an indicator of the asymmetry of the random variable. The kurtosis is an indicator of the ”peakedness” of the distribution: the higher the kurtosis, the more of the variance is due to infrequent extreme deviations. For a Gaussian-distributed noise, skewness and kurtosis are zero.

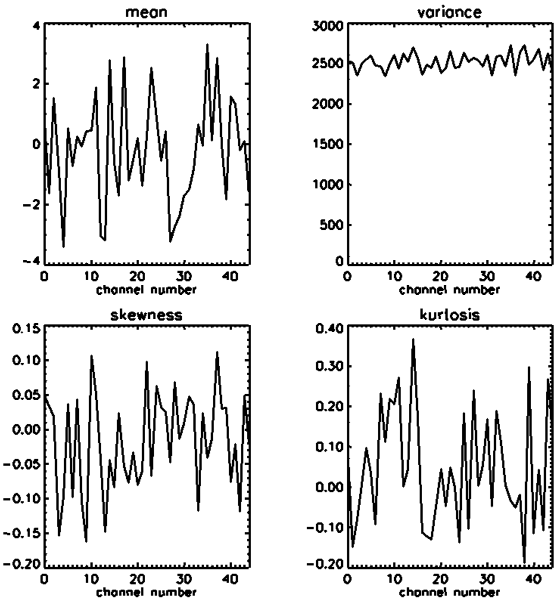

Fig. 3

First four moments about the mean of the random input signal. Top Left: Mean signal as a function of the signal number. Top Right: Second moment, variance. Bottom Left: Third moment, skewness. Bottom Right: Fourth moment, kurtosis.

Here, we shall demonstrate the effect of ARSOS[45] filter on 45 signals of 2048 samples containing simulated Gaussian noise $N(0, 2500)$. Fig. 3 displays the first four moments about the mean for the given signal matrix. Due to the fact that we work with a finite sample size ($N = 2048$), the observed values of the four moments will deviate from the "true" ones – mean=0, variance=2500, skewness=0, kurtosis=0. In fact, the four moments are random variables themselves too, having their own expectation value and variance. Fig. 4 displays the values of the first four moments of the 45×2048 data matrix as a function of filter output channel number.

Note that in our ARSOS filter implementation the first moment of each output channel is automatically set to zero, *i.e.*, a constant value is subtracted from each signal such that its mean becomes zero.

The fact that order-statistics filtering improves the SNR (see Eq. (11)) like conventional signal averaging, can be viewed in Fig. 4. From the variance curve (upper left) in this Figure one can see that the output SNR of the filter depends on the output channel number. The best SNR is theoretically obtained for the median-channel output; the worst SNR is obtained for the minimum-filtered output channel, #1, and maximum-filtered output channel #M.

The skewness as a function of channel number is displayed in the lower left curve. Theoretically, for the output channel numbers lower that the median, the outputs have a negative skewness, for the output channels larger than the median we have a positive skewness. Under the assumption that the variance in the skewness estimation can be approximated from the middle part of the curve, *i.e.*, the values around the median output, skewness differs significantly from zero close to the minimum- and maximum-filtered outputs.

The curve on the lower-right displays the kurtosis as a function of the output channel. Theoretically, also the kurtosis deviates significantly from zero in, and close to, the minimum and maximum output channels. For this particular noise realisation, only the channels close to the minimum filter deviate significantly from zero.

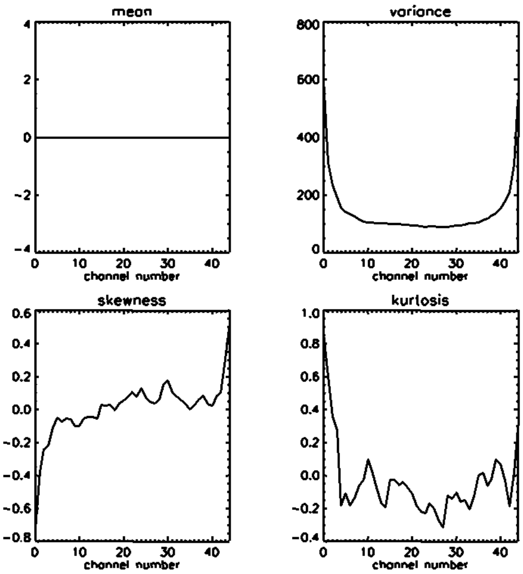

Fig. 4

First four moments about the mean of the ARSOS-filter outputs as a function of the output channel number. Note that the skewness deviates strongly from zero, which means that the pdfs are asymmetric.

IV. Concluding Remarks

This paper deals with the effect of ARSOS-filtering on the pdfs of the output signals. The paper gives analytic expressions for the output pdfs given the shape of the pdf of the input noise. The paper demonstrates that given Gaussian input noise, the pdfs of the output noise are pairwise anti-symmetric. In order to illustrate that output pdfs of order-statistics filtering are non-Gaussian and asymmetric, we compute the first four moments about the mean for an ARSOS[45]-filter.

As a next step, Monte Carlo simulations, including outliers due to patient motion, are needed to quantify the effect of using all channels of an ARSOS filter.

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