Alternative description of the linear canonical integral transformation

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Abstract—**First-optical systems (or ABCD-systems) with a singular submatrix B are considered. Starting with the Iwasawa-type decomposition of a first-order optical system as a cascade of a lens, a magnifier, and an ortho-symplectic system (a system that is both symplectic and orthogonal), a further decomposition of the ortho-symplectic system in the form of a separable fractional Fourier transformer embedded in between two spatial-coordinate rotators is proposed. The resulting decomposition of the entire first-order optical system leads to a representation of the linear canonical integral transformation, which is valid also in the case of a singular submatrix B. Some examples of ABCD-systems with a singular submatrix B are given.**

Keywords—**canonical transformation, Collins integral, symplectic matrix, Iwasawa decomposition**

I. Introduction

Any first-order optical system (or $\mathbf{ABCD}$-system) can be associated with a linear canonical integral transformation, described by Collins integral as long as the submatrix $\mathbf{B}$ is non-singular. To avoid the singular case, Moshinsky and Quesne have shown a decomposition of a symplectic $\mathbf{ABCD}$-matrix with a singular $\mathbf{B}$, as a cascade of two matrices that do not have such a singularity; the way to find these matrices, however, is not easy. In this paper we will show an alternative way to avoid possible difficulties that may arise from a singular submatrix $\mathbf{B}$.

Starting with the Iwasawa-type decomposition of a first-order optical system as a cascade of an ortho-symplectic system (a system that is both symplectic and orthogonal), a magnifier, and a lens, a further decomposition of the ortho-symplectic system is considered for the practically important case that the submatrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ have dimensions 2×2. We propose a decomposition of the ortho-symplectic system in the form of a separable fractional Fourier transformer embedded in between two rotators. The resulting decomposition of the entire first-order optical system then shows a physically attractive way to overcome the singular case in the Collins integral. In particular, we will be able to present the linear canonical integral transformation (whether or not with a singular submatrix $\mathbf{B}$) in the basic form of a separable fractional Fourier transformation; this Fourier transformer then acts on rotated input coordinates, and is followed by a further rotation of the output coordinates, by a magnifier, and by a multiplication with a quadratic phase function (a lens).

II. Lossless first-order optical systems

Any lossless first-order optical system can be described by its ray transformation matrix [1], which relates the position $\mathbf{r}_i$ and direction $\mathbf{q}_i$ of an incoming ray to the position $\mathbf{r}_o$ and direction $\mathbf{q}_o$ of the outgoing ray:

$$\begin{bmatrix}\mathbf{r}_o\\ \mathbf{q}_o\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{r}_i\\ \mathbf{q}_i\end{bmatrix} \equiv \mathbf{T}\begin{bmatrix}\mathbf{r}_i\\ \mathbf{q}_i\end{bmatrix}. \tag{1}$$

The ray transformation matrix $\mathbf{T}$ of such a system is real and symplectic, yielding the relations

$$\begin{aligned}&\mathbf{AB}^t = \mathbf{BA}^t, \quad \mathbf{CD}^t = \mathbf{DC}^t, \quad \mathbf{AD}^t - \mathbf{BC}^t = \mathbf{I},\\ &\mathbf{A}^t\mathbf{C} = \mathbf{C}^t\mathbf{A}, \quad \mathbf{B}^t\mathbf{D} = \mathbf{D}^t\mathbf{B}, \quad \mathbf{A}^t\mathbf{D} - \mathbf{C}^t\mathbf{B} = \mathbf{I}.\end{aligned} \tag{2}$$

Using the submatrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$, and assuming that $\mathbf{B}$ is a non-singular matrix, we can represent the first-order optical system by the Collins integral [2]

$$f_o(\mathbf{r}_o) = \frac{\exp(i\phi)}{\sqrt{\det i\mathbf{B}}}\iint_{-\infty}^{\infty} f_i(\mathbf{r}_i)\exp\left[i\pi\left(\mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{A}\mathbf{r}_i - 2\mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{r}_o + \mathbf{r}_o^t\mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o\right)\right]\mathrm{d}\mathbf{r}_i, \tag{3}$$

where the output amplitude $f_o(\mathbf{r})$ is expressed in terms of the input amplitude $f_i(\mathbf{r})$. The phase factor $\exp(i\phi)$ in

Eq. (3) has been included for completeness, to be able to cope with the optical path length and the metaplectic sign problem; unless absolutely necessary, it will usually be omitted. In the case that $\mathbf{B}$ equals the null matrix, $\mathbf{B} = \mathbf{0}$, Collins integral (3) reduces to

$$f_o(\mathbf{r}) = f_i(\mathbf{A}^{-1}\mathbf{r}) \exp(i\pi\mathbf{r}^t\mathbf{C}\mathbf{A}^{-1}\mathbf{r})/\sqrt{|\det\mathbf{A}|}. \quad (4)$$

The singular case $\det\mathbf{B} = 0$ with $\mathbf{B} \neq \mathbf{0}$, however, is rather difficult to handle.

To treat the singular case, Moshinsky and Quesne [3] showed that any symplectic $\mathbf{ABCD}$-matrix with a singular submatrix $\mathbf{B}$ can be decomposed as

$$\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} = \begin{bmatrix}\mathbf{I} & \mathbf{B}'\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{A}-\mathbf{B}'\mathbf{C} & \mathbf{B}-\mathbf{B}'\mathbf{D}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}, \quad (5)$$

where $\mathbf{B}'$ is a non-singular diagonal matrix and $\det(\mathbf{B}-\mathbf{B}'\mathbf{D}) \neq 0$. After doing so, they could then use the Collins integral (3) for each of the two subsystems in the cascade (5) separately, thus avoiding the singular case. The way to find the diagonal matrix $\mathbf{B}'$, however, is not easy.

In this paper we restrict ourselves to the case that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are 2×2 matrices and we propose a representation of the linear canonical integral transformation in an alternative form that is valid for any ray transformation matrix, whether or not having a singular submatrix $\mathbf{B}$. Our method is based on the Iwasawa decomposition [4, 5], followed by a further decomposition of an ortho-symplectic system into a separable fractional Fourier transformer embedded in between two spatial-coordinate rotators.

III. Iwasawa decomposition of a symplectic matrix

After properly normalizing to dimensionless variables, any symplectic matrix can be decomposed in the Iwasawa form [5, Sections 9.5 and 10.2]

$$\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{G} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{S} & \mathbf{0}\\ \mathbf{0} & \mathbf{S}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{X} & \mathbf{Y}\\ -\mathbf{Y} & \mathbf{X}\end{bmatrix}, \quad (6)$$

where the first matrix represents a lens described by the symmetric matrix

$$\mathbf{G} = -(\mathbf{C}\mathbf{A}^t + \mathbf{D}\mathbf{B}^t)(\mathbf{A}\mathbf{A}^t + \mathbf{B}\mathbf{B}^t)^{-1}, \quad (7)$$

the second matrix represents a magnifier described by the symmetric matrix

$$\mathbf{S} = (\mathbf{A}\mathbf{A}^t + \mathbf{B}\mathbf{B}^t)^{1/2}, \quad (8)$$

and the third matrix represents a so-called ortho-symplectic system – a system that is both orthogonal and symplectic – described by the unitary matrix

$$\mathbf{U} = \mathbf{X} + i\mathbf{Y} = (\mathbf{A}\mathbf{A}^t + \mathbf{B}\mathbf{B}^t)^{-1/2}(\mathbf{A} + i\mathbf{B}). \quad (9)$$

Note that $\mathbf{B} = \mathbf{S}\mathbf{Y}$, and since $\mathbf{S}$ is non-singular, a singularity of $\mathbf{B}$ is only due to the ortho-symplectic system.

IV. The ortho-symplectic system

To treat the ortho-symplectic system in more detail, we remark that if $\mathbf{U}_1$ and $\mathbf{U}_2$ are the unitary representations of two ortho-symplectic matrices $\mathbf{T}_1$ and $\mathbf{T}_2$, then the product $\mathbf{U}_2\mathbf{U}_1$ is the unitary representation of the cascade $\mathbf{T}_2\mathbf{T}_1$. The unitary matrix

$$\mathbf{U}_r(\vartheta) = \mathbf{X}_r(\vartheta) = \begin{bmatrix}\cos\vartheta & \sin\vartheta\\ -\sin\vartheta & \cos\vartheta\end{bmatrix} \quad (10)$$

and its associated ortho-symplectic one

$$\mathbf{T}_r(\vartheta) = \begin{bmatrix}\cos\vartheta & \sin\vartheta & 0 & 0\\ -\sin\vartheta & \cos\vartheta & 0 & 0\\ 0 & 0 & \cos\vartheta & \sin\vartheta\\ 0 & 0 & -\sin\vartheta & \cos\vartheta\end{bmatrix}, \quad (11)$$

cf. Ref. [5, Eq. (10.16)], correspond to a rotator with rotation angle ϑ, which produces a rotation through the angle ϑ both for the spatial variables (x, y) and the spatial-frequency variables (q_x, q_y). The unitary matrix

$$\begin{aligned}\mathbf{U}_f(\gamma_x,\gamma_y) &= \mathbf{X}_f(\gamma_x,\gamma_y) + i\mathbf{Y}_f(\gamma_x,\gamma_y)\\ &= \begin{bmatrix}\exp(i\gamma_x) & 0\\ 0 & \exp(i\gamma_y)\end{bmatrix}\end{aligned} \quad (12)$$

and its associated ortho-symplectic one

$$\mathbf{T}_f(\gamma_x,\gamma_y) = \begin{bmatrix}\cos\gamma_x & 0 & \sin\gamma_x & 0\\ 0 & \cos\gamma_y & 0 & \sin\gamma_y\\ -\sin\gamma_x & 0 & \cos\gamma_x & 0\\ 0 & -\sin\gamma_y & 0 & \cos\gamma_y\end{bmatrix}, \quad (13)$$

cf. Ref. [5, Eq. (10.31)], correspond to a separable fractional Fourier transformer with fractional angles γ_x and γ_y in the x- and the y-direction, respectively, which produces a rotation through γ_x for the space/spatial-frequency combination (x, q_x) and a rotation through γ_y for the space/spatial-frequency combination (y, q_y). We remark that in the one-dimensional case, the ortho-symplectic system that appears in the Iwasawa decomposition (6) is described by the scalar $u = (a + ib)(a^2 + b^2)^{-1/2} = \exp(i\gamma)$, cf. (9), corresponding to a (one-dimensional) fractional Fourier transformer; see also Ref. [6, Section 9.7, in particular Eqs. (9.124)–(9.128)].

We now show that any unitary matrix $\mathbf{U}$ can be represented as a separable fractional Fourier transformer

$\mathbf{U}_f(\gamma_x,\gamma_y)$ embedded in between two rotators $\mathbf{U}_r(\alpha)$ and $\mathbf{U}_r(\beta)$, cf. Ref. [5, Eq. (10.32)]:

$$\mathbf{U} = \mathbf{U}_r(\beta)\,\mathbf{U}_f(\gamma_x,\gamma_y)\,\mathbf{U}_r(\alpha). \tag{14}$$

Note that $\det \mathbf{U}_f(\gamma_x,\gamma_y) = \exp[i(\gamma_x+\gamma_y)]$ and that $\det \mathbf{U}_r(\vartheta) = 1$. We will demonstrate how the four angles α, β, γ_x and γ_y follow from the entries of the matrix $\mathbf{U} = \mathbf{X} + i\mathbf{Y}$:

$$\begin{aligned}
X_{11} &= \cos\alpha\cos\beta\cos\gamma_x - \sin\alpha\sin\beta\cos\gamma_y,\\
X_{12} &= \sin\alpha\cos\beta\cos\gamma_x + \cos\alpha\sin\beta\cos\gamma_y,\\
X_{21} &= -\cos\alpha\sin\beta\cos\gamma_x - \sin\alpha\cos\beta\cos\gamma_y,\\
X_{22} &= -\sin\alpha\sin\beta\cos\gamma_x + \cos\alpha\cos\beta\cos\gamma_y,\\
Y_{11} &= \cos\alpha\cos\beta\sin\gamma_x - \sin\alpha\sin\beta\sin\gamma_y,\\
Y_{12} &= \sin\alpha\cos\beta\sin\gamma_x + \cos\alpha\sin\beta\sin\gamma_y,\\
Y_{21} &= -\cos\alpha\sin\beta\sin\gamma_x - \sin\alpha\cos\beta\sin\gamma_y,\\
Y_{22} &= -\sin\alpha\sin\beta\sin\gamma_x + \cos\alpha\cos\beta\sin\gamma_y.
\end{aligned} \tag{15}$$

We remark the relations

$$\begin{aligned}
\mathbf{U}_r(\vartheta+\pi) &= -\mathbf{U}_r(\vartheta), && (16)\\
\mathbf{U}_f(\gamma_x+\pi,\gamma_y+\pi) &= -\mathbf{U}_f(\gamma_x,\gamma_y), && (17)
\end{aligned}$$

from which we conclude that in the cascade (14) the fractional angles γ_x and γ_y may be safely restricted to the interval $[0,\pi)$, if at the same time we allow at least one of the rotation angles α and β to be in a full interval of length 2π. Moreover, from the relation

$$\mathbf{U}_r(\pi/2)\,\mathbf{U}_f(\gamma_x,\gamma_y)\,\mathbf{U}_r(-\pi/2) = \mathbf{U}_f(\gamma_y,\gamma_x), \tag{18}$$

we conclude that we can always interchange the fractional angles; in other words, we may freely choose the fractional angle γ_x to be the largest. Without loss of generality we may thus choose $0 \le \gamma_y \le \gamma_x < \pi$.

From the two relations

$$\begin{aligned}
\cos(\gamma_x+\gamma_y) &= \Re\{\det\mathbf{U}\},\\
\sin(\gamma_x+\gamma_y) &= \Im\{\det\mathbf{U}\},
\end{aligned} \tag{19}$$

we can determine the sum $\gamma_x+\gamma_y = 2\gamma_1$ (with γ_1 in the interval $0 \le \gamma_1 < \pi$) and from the relation

$$\det\mathbf{X} + \det\mathbf{Y} = \cos(\gamma_x-\gamma_y) \tag{20}$$

we determine the difference $\gamma_x-\gamma_y = 2\gamma_2$ (with γ_2 in the interval $0 \le \gamma_2 < \pi/2$). The two fractional angles γ_x and γ_y are thus completely defined.

To determine the rotation angles α and β, we use the relations

$$\begin{aligned}
X_{11}+X_{22}-Y_{12}+Y_{21} &= 2\cos(\alpha+\beta+\gamma_1)\cos\gamma_2,\\
X_{12}-X_{21}+Y_{11}+Y_{22} &= 2\sin(\alpha+\beta+\gamma_1)\cos\gamma_2,\\
-X_{11}+X_{22}+Y_{12}+Y_{21} &= 2\sin(\alpha-\beta+\gamma_1)\sin\gamma_2,\\
X_{12}+X_{21}+Y_{11}-Y_{22} &= 2\cos(\alpha-\beta+\gamma_1)\sin\gamma_2.
\end{aligned} \tag{21}$$

Note that the case $\gamma_x = \gamma_y$, and thus $\gamma_2 = 0$, is special. In this case we can only determine the sum $\alpha+\beta$ of the two rotation angles. This is obvious, since the cascade (14) now consists of two rotators and an *isotropic* fractional Fourier transformer with a *scalar* matrix $\mathbf{U}(\gamma_x,\gamma_x) = \exp(i\gamma_x)\,\mathbf{I}$; and since the three subsystems commute, we may distribute the total rotation angle $\alpha+\beta$ arbitrarily over the two rotators.

As an example we consider the non-trivial singular case $\gamma_y = 0, \gamma_x \neq 0$, for which $\det\mathbf{B} = 0$ but $\mathbf{B} \neq \mathbf{0}$, and we assume that Y_{11}, Y_{12}, Y_{21}, and Y_{22} do not vanish. From Eq. (15) we conclude directly that $Y_{12}/Y_{11} = Y_{22}/Y_{21} = \tan\alpha$ and $Y_{21}/Y_{11} = Y_{22}/Y_{12} = -\tan\beta$. We recall that at least one rotation angle should be in a full interval of length 2π; choosing rather arbitrarily $\alpha \in [0,\pi)$ and $\beta \in [0,2\pi)$, the angles then appear in the quadrants as shown in Table I. The table is also helpful to solve the apparent π ambiguity for β in the case of vanishing Y values: for instance, to choose between 0 and π if $Y_{21} = Y_{22} = 0$ (using Y_{11} and Y_{12}), and between $\pi/2$ and $3\pi/2$ if $Y_{11} = Y_{12} = 0$ (using Y_{21} and Y_{22}).

TABLE I

THE QUADRANTS FOR $\alpha \in [0,\pi)$ AND $\beta \in [0,2\pi)$, FOR DIFFERENT VALUES OF THE SIGNS OF Y_{11}, Y_{12}, Y_{21}, AND Y_{22}.

Y_{11}	Y_{12}	Y_{21}	Y_{22}	α	β
+	+	−	−	$[0,\pi/2)$	$[0,\pi/2)$
−	+	+	−	$[\pi/2,\pi)$	$[0,\pi/2)$
−	−	−	−	$[0,\pi/2)$	$[\pi/2,\pi)$
+	−	+	−	$[\pi/2,\pi)$	$[\pi/2,\pi)$
−	−	+	+	$[0,\pi/2)$	$[\pi,3\pi/2)$
+	−	−	+	$[\pi/2,\pi)$	$[\pi,3\pi/2)$
+	+	+	+	$[0,\pi/2)$	$[3\pi/2,2\pi)$
−	+	−	+	$[\pi/2,\pi)$	$[3\pi/2,2\pi)$

We conclude that any ortho-symplectic system can be realized as a separable fractional Fourier transformer $\mathbf{U}_f(\gamma_x,\gamma_y)$ embedded in between two rotators $\mathbf{U}_r(\alpha)$ and $\mathbf{U}_r(\beta)$, see Eq. (14).

V. THE FINAL DECOMPOSITION

Using Eqs. (6) and (14), the entire system can finally be decomposed as the cascade

$$\begin{aligned}
\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} &= \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{G} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{S} & \mathbf{0}\\ \mathbf{0} & \mathbf{S}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{X}_r(\beta) & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_r(\beta)\end{bmatrix}\\
&\times \begin{bmatrix}\mathbf{X}_f(\gamma_x,\gamma_y) & \mathbf{Y}_f(\gamma_x,\gamma_y)\\ -\mathbf{Y}_f(\gamma_x,\gamma_y) & \mathbf{X}_f(\gamma_x,\gamma_y)\end{bmatrix}\begin{bmatrix}\mathbf{X}_r(\alpha) & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_r(\alpha)\end{bmatrix}.
\end{aligned} \tag{22}$$

The submatrix $\mathbf{B}$ then takes the form

$$\mathbf{B} = \mathbf{S}\,\mathbf{X}_r(\beta)\,\mathbf{Y}_f(\gamma_x,\gamma_y)\,\mathbf{X}_r(\alpha), \tag{23}$$

and since $\mathbf{S}$ and $\mathbf{X_r}$ are non-singular, the case $\det \mathbf{B} = 0$ arises only for $\det \mathbf{Y}_f(\gamma_x,\gamma_y) = 0$, i.e., for $\sin\gamma_x \sin\gamma_y = 0$, and thus for $\gamma_y = 0$.

The decomposition (22) has a clear physical interpretation. The cascade starts with a rotator $\mathbf{U}_r(\alpha)$ that rotates the coordinate system such that the new axes coincide with the axes of the separable fractional Fourier transformer $\mathbf{U}_f(\gamma_x,\gamma_y)$. Hence, if $f_{\gamma_x}(x_o) = \mathcal{R}^{\gamma_x}[f(x_i)](x_o)$ denotes a one-dimensional fractional Fourier transformation [6–8] with fractional angle γ_x (and $0 \le \gamma_x < \pi$),

$$\mathcal{R}^{\gamma_x}[f(x_i)](x_o) = \begin{cases} \dfrac{\exp(i\gamma_x/2)}{\sqrt{i\sin\gamma_x}} \displaystyle\int_{-\infty}^{\infty} \exp\left[i\pi\frac{(x_o^2+x_i^2)\cos\gamma_x}{\sin\gamma_x}\right] \\ \quad\times \exp\left[-i2\pi\dfrac{x_o x_i}{\sin\gamma_x}\right] f(x_i)\,\mathrm{d}x_i \quad (\gamma_x \ne 0), \\ f(x_o) \quad (\gamma_x = 0), \end{cases} \tag{24}$$

and analogously $f_{\gamma_x,\gamma_y}(\mathbf{r}_o) = \mathcal{R}^{\gamma_\S,\gamma_\dagger}[f(\mathbf{r}_i)](\mathbf{r}_o)$ its two-dimensional separable version with angles γ_x and γ_y, then the input-output relationship for the non-separable fractional Fourier transformer $\mathbf{U}_f(\gamma_x,\gamma_y)\,\mathbf{U}_r(\alpha)$ reads $f_{\gamma_x,\gamma_y}(\mathbf{r}_o) = \mathcal{R}^{\gamma_\S,\gamma_\dagger}[f(\mathbf{X}_r(\alpha)\mathbf{r}_i)](\mathbf{r}_o)$, or in detail:

$$f_{\gamma_x,\gamma_y}(x_o,y_o) = \mathcal{R}^{\gamma_x,\gamma_y}\big[f(x_i\cos\alpha + y_i\sin\alpha, \\ - x_i\sin\alpha + y_i\cos\alpha)\big](x_o,y_o). \tag{25}$$

The separable fractional Fourier transformer $\mathbf{U}_f(\gamma_x,\gamma_y)$ itself is responsible for a possible degeneration of the submatrix $\mathbf{B}$, but such a degeneration has a clear interpretation: it simply means that for one coordinate (or maybe even for both coordinates) the fractional Fourier transformer acts as an identity system. The cascade then continues with the rotator $\mathbf{U}_r(\beta)$, followed by the magnifier $\mathbf{S}$ and the lens $\mathbf{G}$, and brings us to the final input-output relationship for the cascade (22):

$$f_o(\mathbf{r}_o) = \exp(-i\pi\mathbf{r}_o^t\mathbf{G}\mathbf{r}_o)/\sqrt{\det\mathbf{S}} \\ \times \mathcal{R}^{\gamma_x,\gamma_y}\left[f_i\left(\mathbf{X}_r(\alpha)\mathbf{r}_i\right)\right]\left(\mathbf{X}_r(-\beta)\mathbf{S}^{-1}\mathbf{r}_o\right). \tag{26}$$

Equation (26) provides the alternative representation of the linear canonical integral transformation in first-order optical systems, valid for any values of the ray transformation matrix. Note that this equation can also be used for numerical calculation of the canonical transform, using the algorithms developed for the fractional Fourier transformation. [6]

VI. Examples of systems with singular submatrix B

Let us consider some features of symplectic $\mathbf{ABCD}$-matrices with singular submatrix $\mathbf{B}$. First we note that in the case of a singular submatrix $\mathbf{B}$, we have $\mathbf{A} \ne \mathbf{0}$ and $\mathbf{D} \ne \mathbf{0}$. Indeed, if $\mathbf{A} = \mathbf{0}$ or $\mathbf{D} = \mathbf{0}$, symplecticity would imply the relation $\mathbf{C}^t = -\mathbf{B}^{-1}$, which is impossible since $\det\mathbf{B} = 0$.

Several examples corresponding to a nontrivial singular submatrix $\mathbf{B}$ are mentioned below. The simplest example, as discussed above at the end of Section 4, corresponds to the fractional Fourier transformer $\mathbf{U}_f(\gamma_x,\gamma_y)$ with $\gamma_y = 0$ or, more general, $\gamma_y = \pi n$ with integer n. The corresponding unitary matrix then takes the form

$$\begin{aligned}\mathbf{U}_f(\gamma_x,\pi n) &= \begin{bmatrix}\exp(i\gamma_x) & 0\\ 0 & (-1)^n\end{bmatrix} \\ &= \begin{bmatrix}\cos\gamma_x & 0\\ 0 & (-1)^n\end{bmatrix} + i\begin{bmatrix}\sin\gamma_x & 0\\ 0 & 0\end{bmatrix} \\ &\equiv \mathbf{X}_f(\gamma_x,\pi n) + i\mathbf{Y}_f(\gamma_x,\pi n)\end{aligned} \tag{27}$$

and the submatrix $\mathbf{Y}_f(\gamma_x,\pi n)$ is singular. A similar result occurs for $\gamma_x = \pi n$, leading to the singular submatrix $\mathbf{Y}_f(\pi n,\gamma_y)$.

More examples arise from the ortho-symplectic cascade $\mathbf{U} = \mathbf{U}_r(\beta)\,\mathbf{U}_f(\gamma_x,\gamma_y)\,\mathbf{U}_r(\alpha) \equiv \mathbf{X} + i\mathbf{Y}$, see Eq. (14), where a separable fractional Fourier transformer is embedded in between two rotators, by choosing specific values of the rotation angles α and β, and the fractional angles γ_x and γ_y. The choice $\alpha = \pm\pi/2$ and $\beta = 0$, for instance, produces a transform that has been discussed in Ref. [9]. It is described by the unitary matrix

$$\begin{aligned}\mathbf{U}(\gamma_x,\gamma_y) &= \begin{bmatrix}\exp(i\gamma_x) & 0\\ 0 & \exp(i\gamma_y)\end{bmatrix}\begin{bmatrix}0 & \pm1\\ \mp1 & 0\end{bmatrix} \\ &= \begin{bmatrix}0 & \pm\exp(i\gamma_x)\\ \mp\exp(i\gamma_y) & 0\end{bmatrix} \\ &\equiv \mathbf{X}(\gamma_x,\gamma_y) + i\mathbf{Y}(\gamma_x,\gamma_y),\end{aligned} \tag{28}$$

and it is easy to see that for $\gamma_x = \pi n$ or $\gamma_y = \pi n$ (with integer n) the corresponding submatrix becomes singular:

$$\mathbf{Y}(\pi n,\gamma_y) = \begin{bmatrix}0 & 0\\ \mp\sin\gamma_y & 0\end{bmatrix} \\ \text{and}\quad \mathbf{Y}(\gamma_x,\pi n) = \begin{bmatrix}0 & \pm\sin\gamma_x\\ 0 & 0\end{bmatrix}. \tag{29}$$

The choice $\beta = 0$, $\gamma_x = 0$, and $\gamma_y = \pi/2$ in the cascade $\mathbf{U} = \mathbf{U}_r(\beta)\,\mathbf{U}_f(\gamma_x,\gamma_y)\,\mathbf{U}_r(\alpha) \equiv \mathbf{X} + i\mathbf{Y}$, leads to a system with singular submatrices $\mathbf{X}$ and $\mathbf{Y}$ having two vanishing elements:

$$\mathbf{X} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{Y} = \begin{bmatrix} 0 & 0 \\ -\sin\alpha & \cos\alpha \end{bmatrix}. \quad (30)$$

We remark that if a Hermite-Gaussian beam is present at the input of this system, we obtain at the system's output a family of generalized Gaussian beams introduced in Ref. [10] and called Hermite-Laguerre-Gaussian modes, which includes the common Hermite-Gaussian and Laguerre-Gaussian modes as particular cases.

Our final example corresponds to the choice $\alpha = \beta$, $\gamma_x = 0$ and $\gamma_y = \pi/2$, in which case we are led to the singular submatrices

$$\mathbf{X} = \begin{bmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & -\sin^2\alpha \end{bmatrix} \quad \text{and} \quad \mathbf{Y} = \begin{bmatrix} -\sin^2\alpha & \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \cos^2\alpha \end{bmatrix}, \quad (31)$$

with four non-vanishing elements if $\alpha \neq \pi n$ and $\alpha \neq \pi/2 + \pi n$.

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