On Stability Tests for Continuous and Discrete-Time Linear Systems

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Abstract—**In order to ensure the stabiltity of an n-th order linear system there are tests (due to Hurwitz and Schur) to check whether the roots of the denominator polynomials of the transfer functions in the continuous and the discrete-time case are in the left complex half-plane or within the unit circle, respectively. In this contribution, the parallel treatment developed for both cases leads to a simple and insightful proof for the classical stability tests. Instead of looking at the location of the roots of a polynomial as a purely mathematical problem, a systems approach is used that determines whether the covariance matrices of the associated linear systems in state space are positive definite. The result is that the three critical constraints for stability (given by Jury [3]) are simply found from the determinants of the generating matrices for the covariance. These critical conditions, which are a subset of the $n+1$ stability constraints, are sufficient if one starts with a stable system and all parameters are varied, for example in an adaptive environment.**

Keywords—**Stability test, Hurwitz, Jury, Schur.**

I. Introduction

The classical problem of finding criteria with which the roots of a polynomial are located in the left complex half-plane was already solved by Hurwitz in 1895 [2] and is well-documented in many textbooks, see Gantmacher [1] for a good overview. The criteria for the related problem where the roots are inside the unit circle were first given by Schur in 1917 [4] for complex polynomials and later significantly simplified by Jury [3] for real polynomials. Here, we treat the two problems in parallel using a systems approach that leads to a simple proof for the validity of the classical stability tests. Specifically, this approach enables us to give a description of the stability region of a discrete-time linear system in the coefficient space, the closed surface boundary of which is defined by only three equations. We start with relating the two stability problems and introducing the associated linear system descriptions.

II. Continuous versus Discrete-Time

Let us consider the characteristic polynomials

$$b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}\, s + b_n \qquad (1)$$

and

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1}\, z + a_n \qquad (2)$$

of an n-th order continuous-time system in the s-domain, versus an n-th order discrete-time system in the z-domain, both with real coefficients. We can switch from the continuous to the discrete-time case with the bilinear transform

$$s = \frac{z-1}{z+1}, \quad \text{or} \quad z = \frac{1+s}{1-s}. \qquad (3)$$

This transform interrelates the coefficients b_i and a_i, so constraints on b_i can always be translated to constraints on a_i and vice versa. For example, the polynomial in z

$$a_0 z^2 + a_1 z + a_2 \qquad (4)$$

transforms into the second-order polynomial in s

$$a_0(1+s)^2 + a_1(1+s)(1-s) + a_2(1-s)^2, \quad \text{or}$$

$$(a_0+a_1+a_2) + (2a_0 - 2a_2)s + (a_0 - a_1 + a_2)s^2, \qquad (5)$$

so

$$\begin{pmatrix} b_2 \\ b_1 \\ b_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}. \qquad (6)$$

Since it is known that a second-order continuous system is stable iff all three coefficients b_i are positive, we can infer that the corresponding discrete-time system is stable iff

$$a_0 + a_1 + a_2 > 0, \;\; a_0 - a_2 > 0, \;\; a_0 - a_1 + a_2 > 0. \qquad (7)$$

Letting $a_0 = 1$, we have found the well-known stability triangle in the a_1, a_2-plane for a second-order discrete-time system.

Note that we have written (6) with the indices of the b-vector in reverse order. This has the advantage that the matrix which relates b_{2-i} and a_i is involutary (it is its own inverse) apart from a factor $\frac{1}{4}$. So we can also write

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \tfrac{1}{4} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} b_2 \\ b_1 \\ b_0 \end{pmatrix}, \qquad (8)$$

where we can simply ignore the factor $\frac{1}{4}$, since it makes no difference for the roots if all coefficients of a polynomial are multiplied by a common factor. Note that the columns of the matrix, let us denote it $\boldsymbol{T}_2$, are found from the coefficients of the polynomials $(z+1)^{2-j}(z-1)^j$, where $j=0,1,2$. Generalizing, we can relate b_{n-i} and a_i by an $(n+1)\times(n+1)$ transformation matrix $\boldsymbol{T}_n$ with columns indexed from $j=0$ to n found from the coefficients of $(z+1)^{n-j}(z-1)^j$, which will be involutary apart from a factor 2^{-n}. For example, $\boldsymbol{T}_6$ is given by

$$\boldsymbol{T}_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 4 & 2 & 0 & -2 & -4 & -6 \\ 15 & 5 & -1 & -3 & -1 & 5 & 15 \\ 20 & 0 & -4 & 0 & 4 & 0 & -20 \\ 15 & -5 & -1 & 3 & -1 & -5 & 15 \\ 6 & -4 & 2 & 0 & -2 & 4 & -6 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}. \quad (9)$$

Note that the entries satisfy $t_{ij}=t_{i-1,\,j}+t_{i-1,\,j+1}+t_{i,\,j+1}$, so the matrix can be built from its zeroth row and its n-th column starting in the upper right corner. Formally, we can write the entries t_{ij} $(i,j=0,\ldots,n)$ of $\boldsymbol{T}_n$ as

$$t_{ij} = \sum_{k=max(0,\; i+j-n)}^{min(i,j)} (-1)^k \begin{pmatrix} n-j \\ i-k \end{pmatrix} \begin{pmatrix} j \\ k \end{pmatrix}, \quad (10)$$

where $\begin{pmatrix} n \\ k \end{pmatrix}$ denotes the binomial coefficient $\dfrac{n!}{(n-k)!\,k!}$.

We cannot really use the transformation $\boldsymbol{T}_n$ to find the constraints on a_i from the constraints on b_i for any given order n, since the expressions become much more involved than with a more direct approach in the discrete-time case. We can, however, draw some interesting conclusions from this transformation. As another example, let us look at the case $n=3$, so

$$\begin{pmatrix} b_3 \\ b_2 \\ b_1 \\ b_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}. \quad (11)$$

As we already saw with $n=2$, the rows of $\boldsymbol{T}_3$ are the coefficients of four linear inequalities in a_i which are necessary conditions (though not sufficient from $n=3$ upwards) for stability of a third-order discrete system (since positivity of b_i for all i is necessary in the continuous-time case). If we let $a_0=1$, these inequalities are

$$\begin{aligned} 1+a_1+a_2+a_3 &> 0 \\ 3+a_1-a_2-3\,a_3 &> 0 \\ 3-a_1-a_2+3\,a_3 &> 0 \\ 1-a_1+a_2-a_3 &> 0 \end{aligned} \quad (12)$$

which delimit a four-sided pyramid (a pyramid with a triangular base and three triangles as side planes) in the three-dimensional coefficient space $\{\,a_1,a_2,a_3\}$. The four corners or angular points of this pyramid are simply given by the columns of $\boldsymbol{T}_3$ (without the zeroth row, since we excluded a_0 as a degree of freedom), so $\underline{a}=(a_1,a_2,a_3)^T$ is $(3,\,3,\,1)^T$, $(1,-1,-1)^T$, $(-1,-1,1)^T$ and $(-3,\,3,-1)^T$, respectively.

Similarly, for an n-th order discrete system we have $n+1$ linear inequalities in a_i (with $a_0=1$) determined by the rows of $\boldsymbol{T}_n$, which delimit a hyper-pyramid in the n-dimensional coefficient space $\{\,a_1,\ldots,a_n\}$ with $n+1$ angular points given by the columns of $\boldsymbol{T}_n$ ($-$row 0). The actual stability region in $\mathbf{R}^n$ (found from the necessary and sufficient conditions on a_i) is a subspace (with a closed surface boundary) of this hyper-pyramid, which is equivalent to the continuous-time case where the stability region is a subspace of the positive quadrant $b_1,\ldots,b_n>0$. The angular points of the hyper-pyramid are on the boundary of the actual stability region, since they are found from the coefficients of the $n{+}1$ polynomials $(z+1)^{n-j}(z-1)^j$. So we can simply determine the ranges of a_i from the rows of $\boldsymbol{T}_n$. For example, from (9) we can conclude that the coefficients of a sixth-order stable polynomial in z (with $a_0=1$) are within the ranges

$$\begin{array}{ll} -6<a_1<6 & -3<a_2<15 \\ -20<a_3<20 & -5<a_4<15 \\ -6<a_5<6 & -1<a_6<1. \end{array} \quad (13)$$

We will return to the description of the stability region of a discrete-time system and its closed surface boundary in Section V.

III. Covariance Matrix of Direct Forms

The polynomials of (1) and (2) are also the characteristic polynomials associated with the direct form state matrices

$$\boldsymbol{A}_c = \begin{pmatrix} -b_1/b_0 & -b_2/b_0 & \cdots & -b_{n-1}/b_0 & -b_n/b_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad (14)$$

$$\boldsymbol{A}_d = \begin{pmatrix} -a_1/a_0 & -a_2/a_0 & \cdots & -a_{n-1}/a_0 & -a_n/a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

We can implement a continuous system with state matrix $\boldsymbol{A}_c$ as a tapped line of integrators s^{-1} and a discrete system with state matrix $\boldsymbol{A}_d$ as a tapped line of delays z^{-1}. The taps (equal to $-b_i$ and $-a_i$ for $i = 1, \ldots, n$) are fed back to the input summation node and divided by b_0 and a_0, respectively. If we apply an input signal $u(t)$ or $u[n]$ at the summation node, the state equations

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}_c\boldsymbol{x}(t) + \boldsymbol{B}_c u(t), \quad \boldsymbol{x}[n+1] = \boldsymbol{A}_d\boldsymbol{x}[n] + \boldsymbol{B}_d u[n] \tag{15}$$

have $\boldsymbol{B}_c = (1/b_0, 0, \ldots, 0)^T$ and $\boldsymbol{B}_d = (1/a_0, 0, \ldots, 0)^T$.

Next, we determine the covariance matrices $\boldsymbol{K}_c$ and $\boldsymbol{K}_d$ of these two systems, defined as $E(\boldsymbol{x}\boldsymbol{x}^T)$. Given that $E(u^2) = 1$, these follow from the Lyapunov equations

$$-\boldsymbol{A}_c\boldsymbol{K}_c - \boldsymbol{K}_c\boldsymbol{A}_c^T = \boldsymbol{B}_c\boldsymbol{B}_c^T, \tag{16}$$

$$\boldsymbol{K}_d - \boldsymbol{A}_d\boldsymbol{K}_d\boldsymbol{A}_d^T = \boldsymbol{B}_d\boldsymbol{B}_d^T, \tag{17}$$

respectively. These equations are easily understood to be correct. For example, to check the validity of (17), just equate $E(\boldsymbol{x}[n]\boldsymbol{x}^T[n])$ and $E(\boldsymbol{x}[n+1]\boldsymbol{x}^T[n+1])$ and use the state equation (15), since $\boldsymbol{K}_d$ must be independent of time. In general, finding the elements k_{ij} of these matrices requires solving $\frac{1}{2}n(n+1)$ linear equations, since $\boldsymbol{K}_c$ and $\boldsymbol{K}_d$ are symmetric. In case of a Direct Form, however, there are only n different entries in the covariance matrix, since $\boldsymbol{K}_d$ has equal entries on its diagonals and $\boldsymbol{K}_c$ has alternating equal entries on its odd cross-diagonals and zeros on its even cross-diagonals. Specifically, for $n = 3$,

$$\boldsymbol{K}_c = \begin{pmatrix} k_1 & 0 & -k_2 \\ 0 & k_2 & 0 \\ -k_2 & 0 & k_3 \end{pmatrix}, \quad \boldsymbol{K}_d = \begin{pmatrix} k_1 & k_2 & k_3 \\ k_2 & k_1 & k_2 \\ k_3 & k_2 & k_1 \end{pmatrix}. \tag{18}$$

That this is true is easiest to see for the discrete-time case: in a delay-line the variance or power of each state is just passed to the next, whereas the cross-power between two neighboring states does not depend on the place in the line but only on the distance between the neighbors. For the continuous-time case, we need to realize that the cross-power between two states in a line of integrators with input variance $E(u^2) = 1$ is given by

$$E(x_i x_k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(j\omega)^{n-i}(-j\omega)^{n-k}}{p_n(j\omega)p_n(-j\omega)}\, d\omega, \tag{19}$$

where $p_n(s)$ is the characteristic polynomial (1). If $i+k$ is odd (which is true on the even cross-diagonals) integration is over an odd function, yielding zero. If $i+k$ is even, the result alternates between some value and its opposite.

Given that the state matrices are of the form (14) and the corresponding covariance matrices are of the form (18), we can develop the Lyapunov equations (16) and (17) into n linear equations in n unknowns $k_1, \ldots, k_n$. Specifically, in the continuous-time case the equations read as

$$\boldsymbol{C}\tilde{\boldsymbol{k}} = \begin{pmatrix} b_1 & b_3 & b_5 & 0 & 0 \\ b_0 & b_2 & b_4 & 0 & 0 \\ 0 & b_1 & b_3 & b_5 & 0 \\ 0 & b_0 & b_2 & b_4 & 0 \\ 0 & 0 & b_1 & b_3 & b_5 \end{pmatrix} \begin{pmatrix} k_1 \\ -k_2 \\ k_3 \\ -k_4 \\ k_5 \end{pmatrix} = \tfrac{1}{2}\boldsymbol{B}_c \tag{20}$$

for $n = 5$, where $\tilde{\boldsymbol{k}}$ stands for the vector of unknowns with alternating sign and the coefficient matrix $\boldsymbol{C}$ has a logical structure that can easily be continued to arbitrary n. In the discrete-time case we can write equivalently

$$\boldsymbol{D}\boldsymbol{k} = a_0\boldsymbol{B}_d = (1, 0, \ldots, 0)^T, \tag{21}$$

where $\boldsymbol{k} = (k_1, \ldots, k_n)^T$ and the coefficient matrix $\boldsymbol{D}$, though less trivial, again has a logical structure that can be continued to arbitrary n. For example, for $n = 3$, we find

$$\boldsymbol{D} = \begin{pmatrix} a_0^2 - a_1^2 - a_2^2 - a_3^2 & -2(a_1a_2 + a_2a_3) & -2a_1a_3 \\ a_1 & a_0 + a_2 & a_3 \\ a_2 & a_1 + a_3 & a_0 \end{pmatrix}. \tag{22}$$

For any n, the first element is given by $d_{11} = a_0^2 - \sum_{k=1}^n a_k^2$ and its submatrix $\boldsymbol{D}_{11}$ takes on the general form

$$\begin{pmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix} + \begin{pmatrix} a_2 & a_3 & \cdots & a_n \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ a_{n-1} & a_n & \cdots & 0 \\ a_n & 0 & \cdots & 0 \end{pmatrix}, \tag{23}$$

whereas the first row and column are (for $i, j \neq 1$)

$$d_{1j} = -2\sum_{k=0}^{n-j} a_{k+1}a_{k+j} \quad \text{and} \quad d_{i1} = a_{i-1}. \tag{24}$$

IV. Stability Test Based on the Covariance

A well-known theorem from system theory states that a controllable linear system is stable if and only if its covariance matrix is positive definite. So, since a Direct Form is controllable, we only need to demand that $\boldsymbol{K}_c$ and $\boldsymbol{K}_d$ are positive definite to ensure that the roots of (1) and (2) are in the left half-plane or within the unit circle, respectively. To that end, let us solve equations (20) and (21) in order to find $\boldsymbol{k}$, or the entries $k_1, \ldots, k_n$ of $\boldsymbol{K}_c$ and $\boldsymbol{K}_d$ of the form (18). Using Cramer's rule, we find (for $j = 1, \ldots, n$)

$$k_j = \frac{|\boldsymbol{C}_{1j}|}{2b_0 \det \boldsymbol{C}} \quad \text{and} \quad k_j = (-1)^{j-1}\frac{|\boldsymbol{D}_{1j}|}{\det \boldsymbol{D}} \tag{25}$$

for the continuous and the discrete-time case, respectively, where $|\boldsymbol{C}_{1j}|$ and $|\boldsymbol{D}_{1j}|$ are the minors of the first row of $\boldsymbol{C}$

and $\boldsymbol{D}$ (i.e. the determinants of the submatrices that remain after deleting the first row and the j-th column).

A matrix is positive definite if its principal minors are positive. Specifically, for $\boldsymbol{K}_c$ we demand

$$|\,k_n\,| > 0, \quad \begin{vmatrix} k_{n-1} & 0 \\ 0 & k_n \end{vmatrix} > 0, \quad \begin{vmatrix} k_{n-2} & 0 & -k_{n-1} \\ 0 & k_{n-1} & 0 \\ -k_{n-1} & 0 & k_n \end{vmatrix} > 0 \tag{26}$$

et cetera, up to $\det \boldsymbol{K}_c > 0$, whereas for $\boldsymbol{K}_d$ we demand

$$|\,k_1\,| > 0, \quad \begin{vmatrix} k_1 & k_2 \\ k_2 & k_1 \end{vmatrix} > 0, \quad \begin{vmatrix} k_1 & k_2 & k_3 \\ k_2 & k_1 & k_2 \\ k_3 & k_2 & k_1 \end{vmatrix} > 0, \; \ldots \tag{27}$$

et cetera, up to $\det \boldsymbol{K}_d > 0$. Note that with $\boldsymbol{K}_c$ we start in the lower right corner. This will lead to simpler results.

Starting with the $n\times n$-determinant condition, we find

$$\det \boldsymbol{K}_c = \frac{1}{2^n b_0 b_n |\boldsymbol{C}_{nn}|^2}, \tag{28}$$

$$\det \boldsymbol{K}_d = \frac{1}{\left(\sum_{i=0}^{n}(-1)^i\, a_i\right)\left(\sum_{i=0}^{n} a_i\right)\, |\tilde{\boldsymbol{D}}_{11}|^2}, \tag{29}$$

where $|\tilde{\boldsymbol{D}}_{11}|$ stands for the determinant of the matrix

$$\begin{pmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix} - \begin{pmatrix} a_2 & a_3 & \cdots & a_n \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ a_{n-1} & a_n & \cdots & 0 \\ a_n & 0 & \cdots & 0 \end{pmatrix}, \tag{30}$$

which is found from $\boldsymbol{D}_{11}$ in (23) by replacing the plus sign by a minus sign. Although only $b_0 b_n > 0$ is required for (28) to be positive, in practice we demand $b_0 > 0$ (or even $b_0 = 1$), so both b_0 and b_n must necessarily be positive. The same holds for the sum and the alternating sum of a_i. From Section II, we know that these statements are equivalent by just looking at the last and the first row of (11):

$$\begin{aligned} b_0 > 0 \quad &\Longleftrightarrow \quad a_0 - a_1 + a_2 - \ldots + (-1)^n a_n > 0, \\ b_n > 0 \quad &\Longleftrightarrow \quad a_0 + a_1 + a_2 + \ldots + a_n > 0. \end{aligned} \tag{31}$$

Next, staying with the continuous-time case and using (25), the remaining $n-1$ conditions (26) are found to be

$$\frac{\Gamma_{i-1}\,\Gamma_i}{2^{n-i}\,\Gamma_{n-1}\,\Gamma_n} > 0 \tag{32}$$

for $i = 1, \ldots, n-1$, where Γ_i denotes the determinant of the submatrix of $\boldsymbol{C}$ containing its first i rows and columns, and where $\Gamma_0 = 1$, per definition. Note, incidentally, that

$$\Gamma_{n-1} = |\boldsymbol{C}_{nn}| \quad \text{and} \quad \Gamma_n = b_n\,\Gamma_{n-1} = \det \boldsymbol{C}. \tag{33}$$

Note also that $|\boldsymbol{C}_{1n}| = b_0\,\Gamma_{n-2}$, hence $k_n = \frac{1}{2}\Gamma_{n-2}/\Gamma_n$, i.e. the condition $k_n > 0$ corresponds to (32) for $i = n-1$. In all, the conditions (31) and (32) are equivalent to

$$\Gamma_i > 0 \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad b_0 > 0, \tag{34}$$

due the following straightforward inductive reasoning. The denominator of (32) is positive, since $\Gamma_n\Gamma_{n-1} = b_n\Gamma_{n-1}^2$ and $b_n > 0$ from (31). The numerator of (32) for $n = 1$ is Γ_1 which must be positive; this then leads to $\Gamma_2 > 0$ from the numerator for $n = 2$, et cetera, up to $\Gamma_{n-1} > 0$. The final constraint $\Gamma_n > 0$ follows from $\Gamma_n = b_n\Gamma_{n-1}$.

The result (34) is the same as that given by Hurwitz [1], [2] to ensure that the roots of a polynomial are in the left complex half-plane. In the context of the proof given here, we can restate the result as follows.

Theorem 1: A necessary and sufficient condition for the polynomial $b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n$ to have all roots in the left complex half-plane is that the $n\times n$ coefficient matrix $\boldsymbol{C}$ of the form (20) is positive definite with $b_0 > 0$.

Returning to the discrete-time case, we develop the n conditions (27) using (25). The result is found to be

$$\frac{\Delta_i^+\,\Delta_i^-}{\Delta_n^+\,\Delta_n^-} > 0 \tag{35}$$

for $i = 0, \ldots, n-1$, where $\Delta_0^+ = \Delta_0^- = 1$, per definition, and where Δ_i^+ and Δ_i^- are the determinants of the matrices $\boldsymbol{X}_i + \boldsymbol{Y}_i$ and $\boldsymbol{X}_i - \boldsymbol{Y}_i$ with $\boldsymbol{X}_i$ and $\boldsymbol{Y}_i$ defined as

$$\boldsymbol{X}_i = \begin{pmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1} & a_{i-2} & \cdots & a_0 \end{pmatrix}, \tag{36}$$

$$\boldsymbol{Y}_i = \begin{pmatrix} a_{n-i+1} & a_{n-i+2} & \cdots & a_n \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ a_{n-1} & a_n & \cdots & 0 \\ a_n & 0 & \cdots & 0 \end{pmatrix}. \tag{37}$$

Note that $\Delta_{n-1}^+ = |\boldsymbol{D}_{11}|$ and $\Delta_{n-1}^- = |\tilde{\boldsymbol{D}}_{11}|$, whereas

$$\Delta_n^+ = \Delta_{n-1}^- \sum_{i=0}^{n} a_i \quad \text{and} \quad \Delta_n^- = \Delta_{n-1}^- \sum_{i=0}^{n} (-1)^i\, a_i. \tag{38}$$

Note also that $k_1 > 0$ corresponds to (35) for $i = n-1$, so

$$\det \boldsymbol{D} = \frac{\Delta_n^+ \Delta_n^-}{\Delta_{n-1}^-} = \Delta_{n-1}^- \sum_{i=0}^{n} a_i \sum_{j=0}^{n} (-1)^j\, a_j. \tag{39}$$

Since with the two constraints (31) the denominator of (35) is positive, the remaining $n-1$ conditions for stability are

$$\Delta_i^+ \Delta_i^- > 0, \quad \text{for} \quad i = 1, \ldots, n-1, \tag{40}$$

which involves having to determine $2n-2$ determinants. The result can be simplified, however, due to the property

$$\Delta_i^+\Delta_i^- = \tfrac{1}{2}\left(\Delta_{i-1}^+\,\Delta_{i+1}^- + \Delta_{i-1}^-\,\Delta_{i+1}^+\right) \qquad (41)$$

for $i=1,\dots,n-1$, and the following inductive reasoning. For $i=1$, (41) reads as $\Delta_1^+\Delta_1^- = \frac{1}{2}(\Delta_2^- + \Delta_2^+)$, so the first two conditions (40) can be replaced by $\Delta_2^+ > 0$ and $\Delta_2^- > 0$, since the sum and the product of Δ_2^+ and Δ_2^- must both be positive. Next, $\Delta_3^+\Delta_3^- = \frac{1}{2}(\Delta_2^+\Delta_4^- + \Delta_2^-\Delta_4^+)$, so the second pair of conditions (40) can be replaced by $\Delta_4^\pm > 0$, et cetera, up to $\Delta_{n-1}^\pm > 0$ for odd n and $\Delta_n^\pm > 0$ for even n. Note that with (31) the two conditions $\Delta_n^\pm > 0$ are equivalent to the single condition $\Delta_{n-1}^- > 0$ on account of (38), so for any n the number of conditions is still $n-1$. Together with (31), the total number of constraints is $n+1$, the same as in the continuous-time case, cf. (34). In all, we have proven the following result, also given by Jury [3].

Theorem 2: The necessary and sufficient conditions for the polynomial $a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ to have all roots within the complex unit circle are given by

$$\begin{array}{ll} & \sum_{i=0}^{n} a_i > 0, \quad \sum_{i=0}^{n} (-1)^i\, a_i > 0, \\ n \text{ even:} & \Delta_{n-1}^- > 0, \\ & \Delta_i^\pm > 0 \quad \text{for } i = 2,4,\dots,n-2; \\ n \text{ odd:} & \Delta_i^\pm > 0 \quad \text{for } i = 2,4,\dots,n-1. \end{array} \qquad (42)$$

V. Critical Constraints for Stabiltity

In Section II we saw that the boundary of the stability region for a discrete-time system is a closed surface in the n-dimensional coefficient space $\{a_1,\dots,a_n\}$, with $a_0 = 1$. We will now show that for any order n this closed surface has two 'flat' sides (given by two linear equalities) and one warped side (for $n > 2$), given by a nonlinear equality.

From (25) we know that all entries of the covariance matrix have a common denominator which is given by (39) as $\det \boldsymbol{D} = \Delta_{n-1}^- \sum_{i=0}^n a_i \sum_{j=0}^n (-1)^j a_j$. For a stable system (poles inside the unit circle) all three factors of $\det \boldsymbol{D}$ are positive on account of Theorem 2. Specifically, with $a_0 = 1$, these three factors are unity in the origin of the coefficient space $\{a_1,\dots,a_n\}$, which represents a trivial stable system having all poles in the origin of the z-plane and $\boldsymbol{K}_d = \boldsymbol{I}$. Moving away from the stable origin in the coefficient space by varying the parameters a_i, the poles are varied within unit circle as long as $\boldsymbol{K}_d$ does not become infinite. A sign change of any of the three factors in the denominator of $\boldsymbol{K}_d$ corresponds to a pole crossing the unit circle. This means that in an adaptive environment, where the parameters are varied starting from a stable system, we need only demand positive signs in the factored denominator of $\boldsymbol{K}_d$ as conditions to ensure that the systems remains stable. These conditions are called the critical constraints [3], of which there are only three for any system order n. Let us look at the cases $n = 2$ and $n = 3$ as an example.

For $n = 2$ the covariance matrix looks like

$$\boldsymbol{K}_d = \frac{\begin{pmatrix} 1+a_2 & -a_1 \\ -a_1 & 1+a_2 \end{pmatrix}}{(1-a_2)(1+a_1+a_2)(1-a_1+a_2)}, \qquad (43)$$

where setting the three factors in the denominator to zero corresponds to the three sides of the stability triangle in the a_1, a_2-plane. As long as all three factors are positive, the poles are inside the unit circle. Apart from these critical constraints, there are no other conditions for stability resulting from Theorem 2. This is only true for the second-order case. Note that $1-a_2$ changes sign if a complex conjugate pair of poles crosses the unit circle, whereas $1+a_1+a_2$ changes sign if a real pole crosses the unit circle at $z=1$ and $1-a_1+a_2$ does so if a real pole crosses at $z=-1$.

For $n = 3$ the covariance matrix is given by $\boldsymbol{K}_d =$

$$\frac{\begin{pmatrix} 1+a_2-a_1a_3-a_3^2 & -a_1+a_2a_3 & x \\ -a_1+a_2a_3 & 1+a_2-a_1a_3-a_3^2 & -a_1+a_2a_3 \\ x & -a_1+a_2a_3 & 1+a_2-a_1a_3-a_3^2 \end{pmatrix}}{(1-a_2+a_1a_3-a_3^2)(1+a_1+a_2+a_3)(1-a_1+a_2-a_3)}, \qquad (44)$$

where $x = -a_2 + a_1a_3 + a_1^2 - a_2^2$. Positiveness of the three factors in the denominator constitutes the set of critical constraints, whereas from (42) we know that there is a fourth condition for stability, necessary only if we do not start from a stable point in the coefficient space, but are testing the stability of some arbitrary point in this space. This fourth condition is, of course, $\Delta_2^+ > 0$, since Δ_2^- is the first factor in the denominator of (44). Note that Δ_2^+ is on the principal diagonal. So what does the three-dimensional stability region look like? We already know that it is a subspace of the pyramid given by (12). In fact, this pyramid and the actual stability region have two (triangular) sides in common, since the first and last inequality of (12) are two of the critical constraints. The third critical constraint divides the pyramid into two halves, one of which is the actual stability region, whereas for the other $\Delta_2^- < 0$ holds. Specifically, the stability region has two flat sides given by the triangles with angular points $(3,\,3,\,1)^T$, $(1,-1,-1)^T$, $(-1,-1,1)^T$ and $(-3,\,3,-1)^T$, $(1,-1,-1)^T$, $(-1,-1,1)^T$, respectively, and one warped side, given by the quadratic equation $1-a_2+a_1a_3-a_3^2 = 0$. This represents a curved surface in $\mathbf{R}^3$ which shares two (straight) edges with each of the two triangles. A fifth edge of the stability region is the line between the points $(1,-1,-1)^T$ and $(-1,-1,1)^T$, where the triangles meet. Even though the stability region is a closed space, the boundary of which is given by only three equations, it is not sufficient to describe it by the critical constraints only. The problem is that a point outside the pyramid given by (12) can satisfy all three critical constraints, made possible by the fact that Δ_2^-, which is negative in the unstable half of the pyramid, can change sign back when passing its edges. For example, the point $(6,\ 8,\ 2)^T$ meets all critical conditions but is definitely not a stable point, since the polynomial z^3+6z^2+8z+2 has three real roots, two of which are outside the unit circle and, indeed, $\Delta_2^+<0$. To conclude this paragraph on the third-order case, we note two things. First, the fourth stability condition $\Delta_2^+>0$, which is quadratic, can be replaced with the simple linear condition $a_2<3$, given that the critical constraints are met, as some basic algebra will confirm. Secondly, of the four angular points of the stability region two are on all three sides of its boundary, i.e. $(1,-1,-1)^T$ and $(-1,-1,1)^T$.

Looking at the general n-th order discrete-time system, we can state the following. The stability region is a closed part of the coefficient space $\{a_1,\ldots,a_n\}$, the boundary of which has two 'flat' sides given by the linear equations $1+\sum_{i=1}^n a_i = 0$ and $1+\sum_{i=1}^n (-1)^i a_i = 0$, and one warped side (for $n>2$) given by the nonlinear equation $\Delta_{n-1}^- = 0$, which contains the coefficient a_n to the power $n-1$ and which cannot be factored or simplified further. It represents a curved surface in $\mathbf{R}^n$ through all $n+1$ angular points of the stability region found from the coefficients of $(z+1)^{n-j}(z-1)^j$. Of these angular points, $n-1$ are on all three sides of the stability region, the two outliers (the points furthest from the origin) found from the coefficients os $(z\pm 1)^n$ being the exception. From this description we get a fair impression of what the n-th order stability region of a discrete-time system looks like. Writing out the curved surface part of its boundary for $n=4$ yields: $\Delta_3^- = 0 =$

$$1-a_2+a_1a_3-a_3^2-a_1^2a_4+a_1a_3a_4+2a_2a_4-a_2a_4^2-a_4-a_4^2+a_4^3 \quad (45)$$

which indeed contains the five angular points $(4,\ 6,\ 4,\ 1)^T$, $(2,\ 0,-2,-1)^T$, $(0,-2,\ 0,\ 1)^T$, $(-2,\ 0,\ 2,-1)^T$ and $(-4,\ 6,-4,\ 1)^T$. Apart from the three critical constraints $\Delta_3^->0$ and $1\pm a_1+a_2\pm a_3+a_4>0$, there are two other (quadratic) conditions resulting from Theorem 2, given by $1\pm a_3\mp a_1a_4-a_4^2>0$, necessary if we are testing the stability of an arbitrary point in the coefficient space. For $n>4$ the expressions for the nonlinear critical constraint become increasingly large, but the angular points of the stability region are still easy to determine from the matrices $\boldsymbol{T}_n$ given in Section II. For example, the seven columns of (9) minus the zeroth row are the angular points of the stability region for $n=6$.

To conclude, we note that there are, of course, critical constraints in the continuous-time case as well, following from the common denominator of the entries of $\boldsymbol{K}_c$ given by (25). If we take $b_0>0$ there are only two critical constraints, given by $b_n>0$ and $\Gamma_{n-1}>0$, since $\det \boldsymbol{C} = b_n\,\Gamma_{n-1}$ from (33). Starting from a stable point in the coefficient space $\{b_1,\ldots,b_n\}$, e.g. found from the coefficients of $(s+1)^n$, we need only test for sign changes of $b_n>0$ and $\Gamma_{n-1}>0$ when varying the parameters b_i. For example, in the second-order case the constraints $b_2>0$ and $\Gamma_1=b_1>0$ are critical and no other condition results from Theorem 1. For $n=3$, we have $b_3>0$ and $b_1b_2-b_0b_3>0$ as critical constraints, whereas (34) additionally requires $b_1>0$ to exclude the possibility of b_1 and b_2 both being negative. (Remember that we need to stay in the positive quadrant, the equivalent of the pyramid in the discrete-time case.) Finally for $n=4$, $b_4>0$ and $b_1b_2b_3-b_0b_3^2-b_1^2b_4>0$ are critical, while $b_1>0$ and $b_1b_2-b_0b_3>0$ are necessary extra conditions in general. Note that the latter condition can be replaced by $b_3>0$, so only one (nonlinear) constraint remains as long as all parameters are taken positive. (Unfortunately, this is no longer true from $n=5$ upwards.) This then also means that, apart from the condition of positivity of the expression in (45), a fourth-order discrete-time system is stable with the linear conditions $1\pm a_1+a_2\pm a_3+a_4>0$ and $2\pm a_1\mp a_3-2a_4>0$ (equivalent to $b_1, b_3>0$), the latter pair replacing the two quadratic conditions in the previous paragraph.

VI. Conclusion

By looking at the covariance matrices of the associated linear systems, we have gained an insight into the mechanism behind the classical polynomial stability tests, especially into the form and shape of the stability region in the coefficient space. The critical constraints for stability that are of interest from a systems viewpoint are simply found from the determinants of the generating matrices $\boldsymbol{C}$ and $\boldsymbol{D}$, of the form (20) and (22), for finding the covariance.

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