Mixing effects in forced nonlinear LC- Oscillators

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***Abstract* – When an external sinusoidal current is applied to nonlinear self-excited oscillator, and the frequencies of self excitation and external current do not coincide, then the voltage at the external current source may have a component with the frequency that is equal to the difference of self-oscillation and external frequencies. The conditions when it happens are analyzed for single and coupled oscillators. If the effect is obtained in quadrature oscillators then the difference frequency components are also in quadrature. A simple model is provided. It is shown that the model can be reduced to an ideal LC-tank excited by external source. The signal from the tank is supplied to a squarer and a narrow-band filter. The results were verified in simulation and experiments.**

***Index terms* – Mixing, nonlinear oscillators, quadrature oscillators, van der Pol oscillators.**

I. INTRODUCTION

Let us consider an ideal parallel LC-circuit (Fig. 1,a) excited by an external current source, $i = I_m \sin \gamma t$, with $\gamma \neq \omega_0 = 1/\sqrt{LC}$. Assume that the connected in cascade squarer does not load the tank. Then the voltage, v , at the tank may be obtained from the differential equation

$$\frac{1}{L}\int v dt + C\frac{dv}{dt} = I_m \sin \gamma t \ . \qquad (1)$$

Differentiating (1) one obtains

$$\frac{d^2 v}{dt^2} + \omega_0{}^2 v = \frac{\gamma I_m}{C} \cos \gamma t \ . \qquad (2)$$

Solving this equation for zero initial conditions $v(0) = dv/dt\,|_{t=0} = 0$ one finds that

$$v = \frac{\gamma I_m}{C(\omega_0{}^2 - \gamma^2)}(\cos \gamma t - \cos \omega_0 t) \ . \qquad (3)$$

This result can also be rewritten as

$$v = \frac{2\gamma I_m}{C(\omega_0{}^2 - \gamma^2)}\left[\sin\frac{(\omega_0 - \gamma)}{2}t\right]\left[\sin\frac{(\omega_0 + \gamma)}{2}t\right]. \qquad (4)$$

Hence, the voltage v represents the beats (Fig. 1,b). Then, similar to demodulation of balance-modulated signals [1], to obtain the signal of difference frequency one adds a squarer and band-pass filter. If these blocks both are having unity gain, then the output signal, v_o , in steady state operation will be

$$v_o = \frac{\gamma I_m}{C(\omega_0{}^2 - \gamma^2)} \cos(\gamma - \omega_0)t \ . \qquad (5)$$

Thus, the model shown in Fig.1,a may be considered as a new mixer. To the best of our knowledge, this mixer is not described in the literature. We arrived to this model trying to explain mixing effect occurring in experiments with integrated quadrature coupled van der Pol oscillators.

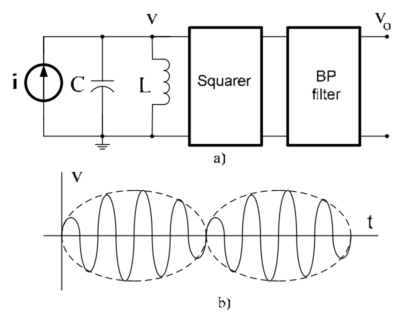

Figure 1. Mixer using externally excited LC-tank

The structure of this paper is the following. Section II explains why the tank of Fig. 1,a may be replaced by a van der Pol LC-oscillator, and what is the difference. Section III explains what happens if one applies the external excitations to two coupled van der Pol oscillators. Section IV explains how the mixing appears when the nonlinearity of the amplitude stabilizing element becomes an asymmetric function of amplitude. Section V gives the circuit where the effect was observed, and simulations confirming the validity of the model. Finally, in section IV we draw some conclusions.

II. TANK REPLACED BY OSCILLATOR

Let us consider a self-excited van der Pol oscillator (Fig. 2,a) which includes a tank with losses and the nonlinear element providing compensation of losses and amplitude stabilization. The circuit will be described by the equation

$$\frac{1}{L}\int v dt + \frac{v}{R} + C\frac{dv}{dt} + i_d = 0 \,, \tag{6}$$

where $i_d = -Mv(1 - Nv^2)$. Here M and N are positive numbers. To demonstrate the method [3] which will be used for search of solutions in this paper, we, first, rewrite this equation in the form

$$\frac{d^2v}{dt^2} - 2(\delta_0 - \delta_2 v^2)\frac{dv}{dt} + \omega_0{}^2 u = 0 \,, \tag{7}$$

where $\delta_0 = \frac{1}{2}\left(M - \frac{1}{R}\right)\frac{1}{C}$ and $\delta_2 = \frac{3MN}{2C}$. We assume that the self-sustained free oscillation is

$$v_s = A\cos\omega t \tag{8}$$

(the zero initial phase is chosen for convenience).To find the amplitude, A , and frequency, ω , we rewrite (7) as

$$\frac{d^2v}{dt^2} + \omega^2 v = (\omega^2 - \omega_0{}^2)v + 2(\delta_0 - \delta_2 v^2)\frac{dv}{dt} \;. \tag{9}$$

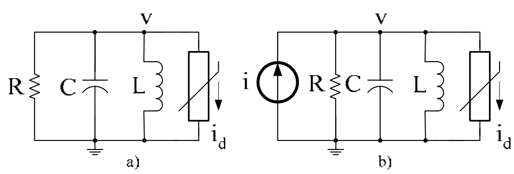

Figure 2. Free and forced van der Pol oscillators

Substituting (8) in the right side of (9) one obtains

$$\begin{aligned} &\frac{d^2v}{dt^2} + \omega^2 v = (\omega^2 - \omega_0{}^2)A\cos\omega t \\ &+2A\omega(\delta_0 - \frac{\delta_2 A^2}{4})\sin\omega t + \ldots \end{aligned} \tag{10}$$

Here the dots denote the terms including higher harmonics. In steady state oscillations the solution (8) makes the left side of (10) equal to zero. Hence, this solution will be possible if the coefficients before $\cos\omega t$ and $\sin\omega t$ terms in (10) are equal to zero as well. This gives

$$\omega = \omega_0; \qquad A = \pm 2\sqrt{\delta_0 / \delta_2} \tag{11}$$

These results are well known [3], and the same method will be used now to find the forced solution (Fig. 2,b). The equation

$$\frac{1}{L}\int v dt + \frac{v}{R} + C\frac{dv}{dt} + i_d = i \;, \tag{12}$$

where $i = I_m \sin\gamma t$, is rewritten as

$$\begin{aligned} &\frac{d^2v}{dt^2} + \gamma^2 v = (\gamma^2 - \omega_0{}^2)v \\ &+2(\delta_0 - \delta_2 v^2)\frac{dv}{dt} + \frac{\gamma I_m}{C}\cos\gamma t \end{aligned} \tag{13}$$

and the forced solution of (13) is searched in the form

$$v_f = A\cos\gamma t - B\sin\gamma t = D\cos(\gamma t + \varphi) \;. \tag{14}$$

Substituting (14) in the right side of (13) one find that

$$\left\{ \begin{aligned} \frac{d^2v}{dt^2} + \gamma^2 v = &\begin{bmatrix} (\gamma^2 - \omega_0{}^2)A \\ -2\gamma B(\delta_0 - \frac{1}{4}\delta_2(A^2 + B^2)) \\ +\frac{\gamma I_m}{C} \end{bmatrix}\cos\gamma t \\ + &\begin{bmatrix} -2(\gamma^2 - \omega_0{}^2)B \\ -2\gamma A\left(\delta_0 - \frac{1}{4}\delta_2(A^2 + B^2)\right) \end{bmatrix}\sin\gamma t + \ldots \end{aligned} \right. \tag{15}$$

Then, to find A and B one has two equations

$$\left\{ \begin{aligned} &(\gamma^2 - \omega_0{}^2)A - 2\gamma B\left(\delta_0 - \frac{\delta_2 D^2}{4}\right) + \frac{\gamma I_m}{C} = 0 \\ &(\gamma^2 - \omega_0{}^2)B - 2\gamma A\left(\delta_0 - \frac{\delta_2 D^2}{4}\right) = 0 \end{aligned} \right. \tag{16}$$

If the external excitation current has I_m and γ satisfying the condition

$$2\sqrt{\frac{\delta_2}{\delta_0}} = \frac{\gamma I_m}{C(\omega_0{}^2 - \gamma^2)} \;, \tag{17}$$

then $A = 2\sqrt{\delta_0 / \delta_2}$ and $B = 0$ satisfy the system (16). Hence, for this particular excitation (this is the difference between linear and nonlinear circuits) one obtains

$$v = v_s + v_f = \frac{\gamma I_m}{C(\omega_0{}^2 - \gamma^2)}(\cos\gamma t \pm \cos\omega_0 t) \,, \tag{18}$$

which is similar to (3). The same A and B are also obtained if γ is far from ω_0 (in the following we assume that this condition is satisfied) and the first terms in (16) are dominating the second ones. Hence, v , in the case of using the oscillator instead of the tank, will be the beats as well, as soon as the frequency of external excitation is reasonably far from the frequency of self-sustained oscillations. The difference is only that the phase of the self-sustained oscillations is independent on the phase of forced oscillation. If the squarer and narrow-band filter are connected to the circuit of Fig. 2,b the filter output signal will be also including the component (5).

It is easy to verify that substitution of (18) in (13) results in the modified imbalance of the third harmonics only. But they were discarded from consideration from the very beginning.

III. TANK REPLACED BY COUPLED OSCILLATORS

Let us consider now the case when the tank is replaced by two coupled oscillators. Assume that two identical van der Pol oscillators (Fig. 3,a) are coupled via controlled transconductances $i_{s1} = gv$ and $i_{s2} = -gu$. To find the self-sustained synchronous solution one has the system

$$\begin{cases} \dfrac{d^2u}{dt^2} - 2(\delta_0 - \delta_2 u^2)\dfrac{du}{dt} + \omega_0{}^2 u + \alpha\dfrac{dv}{dt} = 0 \\ \dfrac{d^2v}{dt^2} - 2(\delta_0 - \delta_2 v^2)\dfrac{dv}{dt} + \omega_0{}^2 v - \alpha\dfrac{du}{dt} = 0 \end{cases} \quad (19)$$

Here $\alpha = g/C$. Then, the solution of this system should have the form

$$\begin{cases} u = A\cos\omega t \\ v = B\sin(\omega t - \varphi) \end{cases} \quad (20)$$

Here ω is the frequency of the synchronous oscillations, and φ is the quadrature phase error.

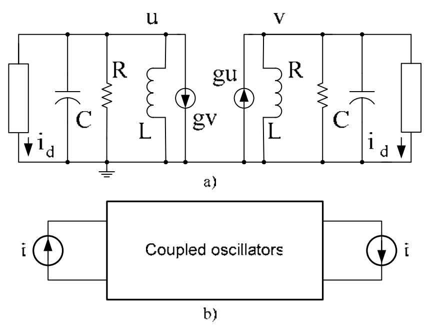

Figure 3. Free (a) and forced (b) oupled oscillators

Using the previously described approach one can easily find that for the synchronous oscillations

$$A = 2\sqrt{\frac{\delta_0 + \dfrac{\alpha}{2}\dfrac{B}{A}\sin\varphi}{\delta_2}}; \quad B = 2\sqrt{\frac{\delta_0 - \dfrac{\alpha}{2}\dfrac{A}{B}\sin\varphi}{\delta_2}}, \quad (21)$$

and the frequency of these oscillations can be found from the equation

$$(\omega^2 - \omega_0{}^2)^2 = \alpha^2\omega^2\cos^2\varphi. \quad (22)$$

For $\varphi = 0$ (quadrature oscillations) $A = B = 2\sqrt{\delta_0/\delta_2}$ and

$$\omega_{1,2} = \sqrt{\omega_0 + \frac{\alpha^2}{4}} \pm \frac{\alpha}{2} \approx \omega_0 \pm \frac{\alpha}{2} \quad (23)$$

The system of coupled oscillators usually oscillates at the lowest frequency, $\omega \approx \omega_0 - \alpha/2 \approx \omega_0$. If the frequency of external excitation (Fig. 3,b) is reasonably far from this frequency, then the voltages forced by this excitation may be found from the system

$$\begin{cases} \dfrac{d^2u}{dt^2} + \omega_0{}^2 u \approx \dfrac{\gamma I_m}{C}\cos\gamma t \\ \dfrac{d^2v}{dt^2} + \omega_0{}^2 v \approx -\dfrac{\gamma I_m}{C}\cos\gamma t \end{cases} \quad (24)$$

This gives

$$u \approx \frac{\gamma I_m}{C(\omega_0{}^2 - \gamma^2)}\cos\gamma t; \quad v \approx -\frac{\gamma I_m}{C(\omega_0{}^2 - \gamma^2)}\cos\gamma t \quad . \quad (25)$$

If the condition (17) is satisfied again then the full solution will be

$$\begin{cases} u \approx \dfrac{\gamma I_m}{C(\gamma^2 - \omega_0{}^2)}(\cos\omega t + \cos\gamma t) \\ = \dfrac{2\gamma I_m}{C(\gamma^2 - \omega_0{}^2)}\cos\dfrac{\omega+\gamma}{2}t\cos\dfrac{\omega-\gamma}{2}t \\ v \approx \dfrac{\gamma I_m}{C(\gamma^2 - \omega_0{}^2)}(\cos\omega t - \cos\gamma t) \\ = \dfrac{2\gamma I_m}{C(\gamma^2 - \omega_0{}^2)}\sin\dfrac{\omega+\gamma}{2}t\sin\dfrac{\omega-\gamma}{2}t \end{cases} \quad (25)$$

Hence, in the obtained beats the high-frequency fillings as well as the low-frequency envelopes are in quadrature.
The substitution of solutions (25) in the full equations

$$\begin{cases} \dfrac{d^2u}{dt^2} - 2(\delta_0 - \delta_2 u^2)\dfrac{du}{dt} + \omega_0{}^2 u + \alpha\dfrac{dv}{dt} = \dfrac{\gamma I_m}{C}\cos\gamma t \\ \dfrac{d^2v}{dt^2} - 2(\delta_0 - \delta_2 v^2)\dfrac{dv}{dt} + \omega_0{}^2 v - \alpha\dfrac{du}{dt} = -\dfrac{\gamma I_m}{C}\cos\gamma t \end{cases} \quad (26)$$

shows that the discarded imbalance of the higher harmonics includes, besides the frequencies 3γ and 3ω, the combination frequencies $2\gamma \pm \omega$ and $2\omega \pm \gamma$.

IV. NONSYMMETRIC AMPLITUDE STABILIZATION

The mixing effect appears inadvertently if the oscillator(s) has an asymmetric characteristic of the amplitude stabilization nonlinearity. The single forced van der Pol oscillator will be described in this case by the equation

$$\frac{d^2v}{dt^2} - 2(\delta_0 + \delta_1 v - \delta_2 v^2)\frac{dv}{dt} + \omega_0{}^2 u = \frac{\gamma I_m}{C}\cos\gamma t \quad (27)$$

Here the coefficient δ_1 depends on the particular design of the amplitude stabilization circuit. In case of most commonly used for this purpose flip-flops its value is defined by the mismatch of transistor geometries, and the threshold mismatch. The inclusion of the term $\delta_1 v\dfrac{dv}{dt}$ results in an asymmetric wave shape. We will consider that this effect is small. To evaluate the effect more precisely, for example, for the case of solution given by (18), we substitute (18) in the right side of the equation

$$\frac{d^2v}{dt^2} + \omega_0{}^2 v = 2(\delta_0 + \delta_1 v - \delta_2 v^2)\frac{dv}{dt} + \frac{\gamma I_m}{C}\cos\gamma t \quad (28)$$

and find the term

$$2\delta_1 v\frac{dv}{dt} = 4\delta_1\frac{\delta_0}{\delta_2}(\gamma - \omega_0)\sin(\gamma - \omega_0)t + ... \quad (29)$$

Considering this term as an external force for the oscillator running at the self-sustained frequency ω_0 we write the equation

$$\frac{d^2 v}{dt^2} + \omega_0{}^2 v = 4\delta_1 \frac{\delta_0}{\delta_2}(\gamma - \omega_0)\sin(\gamma - \omega_0)t\ , \qquad (30)$$

and obtain that the forced solution of this equation is

$$v_d = \frac{4\delta_0\delta_1(\gamma - \omega_0)}{\delta_0\gamma\left(2\omega_0 - \gamma\right)}\sin(\gamma - \omega_0)t\ . \qquad (31)$$

The amplitude $F_d = \dfrac{4\delta_0\delta_1(\gamma - \omega_0)}{\delta_0\gamma\left(2\omega_0 - \gamma\right)}$ represents the mixing difference frequency tone that appears in the oscillator spectrum.

When the forced solution is found from the system (16) one may consider that the solution is not much different from

$$v = D\cos(\gamma t + \varphi) - 2\sqrt{\delta_0/\delta_2}\cos\omega_0 t \qquad (32)$$

Using the same approach one finds that in this case the amplitude of the mixing component is equal to

$$F_d = 2\delta_1 D\sqrt{\frac{\delta_0}{\delta_2}}\frac{(\gamma - \omega_0)}{\gamma(2\omega_0 - \gamma)} \qquad (33)$$

Substituting $\omega \approx \omega_0 - \alpha/2$ instead of ω_0 in (31) one can evaluate the amplitudes of mixing components in each constituent oscillator of the quadrature oscillator.

V. Experimental and Simulation Results

The mixing effect was first discovered in the quadrature coupled LC-oscillator shown in Fig. 4. The constituent oscillators in this case are Robinson, and not van der Pol oscillators. Yet, the analysis of Robinson oscillators is more cumbersome but does not involve any new concepts and gives the results sufficiently close to van der Pol model. Thus, using van der Pol model we were able to obtain tractable results that could be verified in simulations.

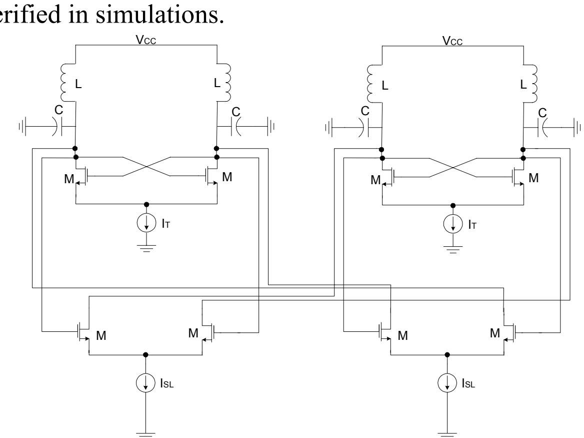

Figure 4. Quadrature LC-oscillator used in experiments and simulations

The oscillators were designed to run at the frequency of $f_0 = 2.2$ GHz. The mixing effect was observed when I_{SL} included an RF component with the frequency of $f_\gamma = 2$ GHz.

The simulations confirmed that the envelopes of the output signals are in quadrature, and, to obtain the mixing effect, the waveform should have visible distortions due to an asymmetry in the amplitude control circuits. These results are not shown due to insufficient space. Finally, Fig. 5 shows the spectrum of output signal. One can see the oscillator signal (f_0=2.2 GHz), the RF signal (f_{RF}=2 GHz), the difference signal ($f_0 - f_{RF}$=0.2 GHz), and the traces of the combination harmonics $2f_0 - f_\gamma = 2.6$ GHz and $2f_\gamma - f_0$=2 GHz. In the range of low frequencies, one can see the traces of the signals with the frequencies of 2($f_0 - f_\gamma$)=0.4 GHz and 3($f_0 - f_\gamma$)=0.6 GHz. These harmonics would appear in calculations if we decided that the representation including oscillator signal, RF and the difference frequency components is not sufficient and calculated more harmonics in the second round of calculations.

VI. Conclusions

The proposed theory and subsequent simulations completely explains the mixing effect when the signal of RF frequency is applied to a nonlinear oscillator (or coupled oscillators). In addition, we introduced a new conceptual model of a mixer than can be used to develop new mixer circuits.

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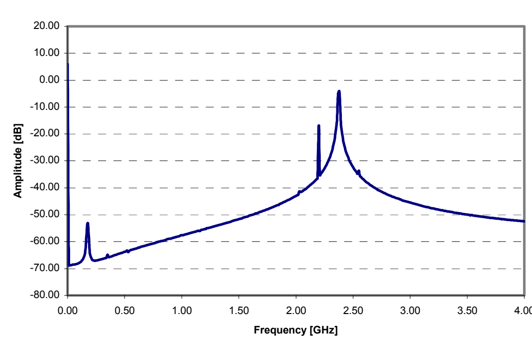

Figure 5. Quadrature LC-oscillator simulated spectrum